# Math 303: Quiz # 1

**Spring 2020**

- Write your name and Student ID number in the space provided below and sign.

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|---|---|
| **Name, Last Name:** | |
| **ID Number:** | |
| **Signature:** | |

- You have One hour.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (10 points) Find all possible values of the real and imaginary parts of $\tan^{-1}(2-i)$. You are expected to simplify your response as much as possible.

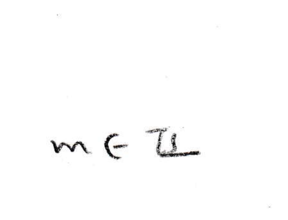
$u := \tan^{-1}(2-i) \Rightarrow \tan(u) = 2-i \Rightarrow \frac{\sin u}{\cos u} = 2-i$

$\Rightarrow \frac{\frac{e^{iu}-e^{-iu}}{2i}}{\frac{e^{iu}+e^{-iu}}{2}} = 2-i \Rightarrow e^{iu}-e^{-iu} = i(2-i)(e^{iu}+e^{-iu})$

$= (1+2i)(e^{iu}+e^{-iu})$

$\Rightarrow (1-1-2i)e^{iu} = (1+2i+1)e^{-iu}$

$\Rightarrow e^{2iu} = \frac{2(1+i)}{-2i} = -(-i+1) = -1+i$




Let $\boxed{x := \mathrm{Re}(u)}$, $\boxed{y := \mathrm{Im}(u)}$

$\Rightarrow e^{2i(x+iy)} = -1+i = \sqrt{2}\,e^{\frac{3\pi i}{4}}$

$\Rightarrow e^{-2y}e^{2ix} = \sqrt{2}\,e^{\frac{3\pi i}{4}}$

$\Rightarrow e^{-2y} = \sqrt{2}$ and $2x = \frac{3\pi}{4} + 2\pi m$ $\quad m \in \mathbb{Z}$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$-2y = \ln\sqrt{2} \Rightarrow \boxed{y = -\frac{\ln 2}{4}}$ $\qquad$ $\boxed{x = \frac{3\pi}{8} + \pi m, \quad m \in \mathbb{Z}}$

**2** (15 points) Find all complex solutions of the equation $\cosh(2z) = 2\cosh(z)$.
Hint: This equation has two real and infinitely many ~~imaginary~~ complex solutions.

$$\Rightarrow \frac{e^{2z}+e^{-2z}}{2} = e^{z}+e^{-z}$$

$$\Rightarrow (e^{z}+e^{-z})^2 - 2 = 2(e^{z}+e^{-z})$$

Let $w := e^{z}+e^{-z} \leadsto w^2 - 2w - 2 = 0$

$$\Rightarrow w = 1 \pm \sqrt{1+2} = 1 \pm \sqrt{3}$$

$$\Rightarrow e^{z}+e^{-z} = 1\pm\sqrt{3} \qquad \text{Let } u := e^{z}$$

$$\Rightarrow u + \frac{1}{u} = 1\pm\sqrt{3} \Rightarrow u^2 - (1\pm\sqrt{3})u + 1 = 0$$

For $w = 1+\sqrt{3}$ : $u^2 - (1+\sqrt{3})u + 1 = 0$

$$\Rightarrow u = \frac{1}{2}\Big[1+\sqrt{3} \pm \sqrt{\underbrace{(1+\sqrt{3})^2 - 4}_{1+3+2\sqrt{3}}}\Big] = \frac{1}{2}\left[1+\sqrt{3} \pm \sqrt{2\sqrt{3}}\right] \in \mathbb{R}^+$$

$$\Rightarrow \boxed{z = \ln\left\{\frac{1}{2}\left[1+\sqrt{3} \pm \sqrt{2\sqrt{3}}\right]\right\} + 2\pi i m, \quad m \in \mathbb{Z}}$$

For $w = 1-\sqrt{3}$ : $u^2 - (1-\sqrt{3})u + 1 = 0$

$$\Rightarrow u = \frac{1}{2}\Big[1-\sqrt{3} \pm \sqrt{\underbrace{(1-\sqrt{3})^2 - 4}_{-2\sqrt{3}}}\Big] = \frac{1}{2}\left[1-\sqrt{3} \pm i\sqrt{2\sqrt{3}}\right]$$

$$\Rightarrow z = \ln\left\{\frac{1}{2}\left[1-\sqrt{3} \pm i\sqrt{2\sqrt{3}}\right]\right\}$$

$$|u| = \frac{1}{2}\sqrt{(1-\sqrt{3})^2 + 2\sqrt{3}} = 1 \qquad \operatorname{Arg}(u) = \tan^{-1}\left(\pm\frac{\sqrt{2\sqrt{3}}}{1-\sqrt{3}}\right)$$

$$= \pm \tan^{-1}\left(\frac{\sqrt{2\sqrt{3}}}{1-\sqrt{3}}\right)$$

$$\rightarrow \boxed{z = i\left[\pm \tan^{-1}\left(\frac{\sqrt{2\sqrt{3}}}{\sqrt{3}-1}\right) + 2\pi n\right], \quad n \in \mathbb{Z}}$$

**3** (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $f(x, y) := x^y$, $p$ denotes the point $(e, 2)$ in $\mathbb{R}^2$, and $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors along the $x$- and $y$-axis in $\mathbb{R}^2$, respectively. Find a real number $\alpha$ such that the directional derivative of $f$ along the vector $\mathbf{v} := \mathbf{i} + \alpha\mathbf{j}$ vanishes at $p$.

$$f(x,y) = x^y = e^{y \ln x}$$

$$\frac{\partial f}{\partial x} = \frac{y}{x} e^{y \ln x}, \qquad \frac{\partial f}{\partial y} = \ln x \, e^{y \ln x}$$

$$\frac{\partial f}{\partial x}(e,2) = \frac{2}{e} e^{2 \ln e} = \frac{2e^2}{e} = 2e$$

$$\frac{\partial f}{\partial y}(e,2) = \ln e \; e^{2 \ln e} = e^2$$

$$\vec{\nabla} f(e,2) = 2e\,\vec{i} + e^2\,\vec{j}$$

$$\hat{v} := \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + \alpha\vec{j}}{\sqrt{1+\alpha^2}}$$

$$0 = (D_{\hat{v}} f)(e,2) = \hat{v} \cdot \vec{\nabla} f(e,2)$$

$$= \frac{1}{\sqrt{1+\alpha^2}} (\vec{i} + \alpha\vec{j}) \cdot (2e\,\vec{i} + e^2\,\vec{j})$$

$$= \frac{1}{\sqrt{1+\alpha^2}} (2e + \alpha e^2) \Rightarrow \boxed{\alpha = -\frac{2}{e}}$$

**4** (10 points) Use the Taylor series expansion of the function $f(x,y) = \dfrac{\ln x}{\ln y}$ about the point $(1,2)$ to approximate it with a quadratic polynomial of the form

$$c_0 + c_1(x-1) + c_2(y-2) + c_3(x-1)^2 + c_4(x-1)(y-2) + c_5(y-2)^2.$$

Determine the values of the coefficients $c_0, c_1, c_2, c_3$, and $c_4$ of this polynomial.

$$f(1,2) = 0$$

$$\frac{\partial f}{\partial x} = \frac{1}{x \ln y} \Rightarrow \frac{\partial f}{\partial x}(1,2) = \frac{1}{\ln 2}$$

$$\frac{\partial f}{\partial y} = \ln x \left[ - \frac{\frac{1}{y}}{(\ln y)^2} \right] = - \frac{\ln x}{y \ln^2 y} \Rightarrow \frac{\partial f}{\partial y}(1,2) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = - \frac{1}{x^2 \ln y} \Rightarrow \frac{\partial^2 f}{\partial x^2}(1,2) = - \frac{1}{\ln 2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{x}\left( - \frac{\frac{1}{y}}{\ln^2 y} \right) = - \frac{1}{x y \ln^2 y} \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(1,2) = - \frac{1}{2 \ln^2 2}$$

$$\frac{\partial^2 f}{\partial y^2} = - \ln x \left[ - \frac{\ln^2 y + 2y \ln y \left(\frac{1}{y}\right)}{y^2 \ln^4 y} \right] = \frac{\ln x (\ln y + 2)}{y^2 \ln^3 y}$$

$$\frac{\partial^2 f}{\partial y^2}(1,2) = 0$$

$$f(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2) +$$
$$\frac{1}{2}\left[ \frac{\partial^2 f}{\partial x^2}(1,2)(x-1)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(1,2)(x-1)(y-2) + \right.$$
$$\left. \frac{\partial^2 f}{\partial y^2}(1,2)(y-2)^2 \right] + \ldots$$
$$= \frac{1}{\ln 2}(x-1) + \frac{1}{2}\left[ - \frac{1}{\ln 2}(x-1)^2 - \frac{1}{\ln^2 2}(x-1)(y-2) \right] + \ldots$$

So $c_0 = 0$, $c_1 = \dfrac{1}{\ln 2}$, $c_2 = 0$,

$c_3 = - \dfrac{1}{2 \ln 2}$, $c_4 = - \dfrac{1}{2 \ln^2 2}$, $c_5 = 0$.