Math 303: Quiz # 2

Spring 2020

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 75 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (10 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be functions with continuous second partial derivatives. Use the properties of the Levi Civita symbol ϵ_{ijk} to express $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g)$ in the form, $\alpha(p + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} g + \beta(q + \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} f$, where α and β are real numbers, p and q are scalar functions, and for every vector-valued functions $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$, $(\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} := \sum_{j=1}^3 F_j \partial_j \mathbf{G}$, and $\partial_j := \frac{\partial}{\partial x_j}$. Find α, β, p and q.

Warning: Solutions not using the properties of ϵ_{ijk} will not be given any credit.

$$[\vec\nabla \times (\vec\nabla f \times \vec\nabla g)]_i = \sum_{j,k=1}^{3} \epsilon_{ijk}\, \partial_j (\vec\nabla f \times \vec\nabla g)_k$$

$$= \sum_{j,k,l,m=1}^{3} \epsilon_{ijk}\, \partial_j [\epsilon_{klm} (\partial_l f)(\partial_m g)]$$

$$= \sum_{j,k,l,m=1}^{3} \epsilon_{kij}\, \epsilon_{klm} [(\partial_j \partial_l f)(\partial_m g) + (\partial_l f)(\partial_j \partial_m g)]$$

$$= \sum_{j,l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) [\ ''\ ]$$

$$= \sum_{j=1}^{3} [(\partial_j \partial_i f)(\partial_j g) + \partial_i f (\partial_j \partial_j g) - (\partial_j \partial_j f)(\partial_i g) - (\partial_j f)(\partial_j \partial_i g)]$$

$$= (\vec\nabla g \cdot \vec\nabla)\, \partial_i f + (\nabla^2 g)\, \partial_i f - (\nabla^2 f)(\partial_i g) - (\vec\nabla f \cdot \vec\nabla)\, \partial_i g$$

$$\Rightarrow \vec\nabla \times (\vec\nabla f \times \vec\nabla g) = -(\nabla^2 f + \vec\nabla f \cdot \vec\nabla)\vec\nabla g + (\nabla^2 g + \vec\nabla g \cdot \vec\nabla)\vec\nabla f$$

So $\boxed{\alpha = -1}$, $\boxed{\beta = 1}$, $\boxed{p = \nabla^2 f}$, $\boxed{q = \nabla^2 g}$

2 (10 points) Find all real numbers a such that the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) := x^2 + axy + 2y^2 + 2z^2 - x + 2y - 4z$ has a local minimum in $\mathbb{R}^3$.

$$f_x = 2x + ay - 1 = 0$$
$$f_y = ax + 4y + 2 = 0$$

$$\boxed{\begin{aligned} 2x + ay &= 1 \\ ax + 4y &= -2 \end{aligned}}$$

$$f_z = 4z - 4 = 0 \quad \Rightarrow \quad \boxed{z = 1}$$

$$\Rightarrow \quad \boxed{x = \frac{\det\begin{bmatrix} 1 & a \\ -2 & 4 \end{bmatrix}}{\det\begin{bmatrix} 2 & a \\ a & 4 \end{bmatrix}} = \frac{4 + 2a}{8 - a^2}}$$

$$\boxed{y = \frac{\det\begin{bmatrix} 2 & 1 \\ a & -2 \end{bmatrix}}{\det\begin{bmatrix} 2 & a \\ a & 4 \end{bmatrix}} = \frac{-4 - a}{8 - a^2}}$$

There is a local extremum point if $a \neq \pm 2\sqrt{2}$.

$f_{xx} = 2$, $f_{xy} = a$, $f_{xz} = 0$

$f_{yy} = 4$, $f_{yz} = 0$, $f_{zz} = 4$

Hessian: $\mathbb{H} = \begin{bmatrix} 2 & a & 0 \\ a & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Eignvalues of $\mathbb{H}$ are 4 and eignvalues of $\begin{bmatrix} 2 & a \\ a & 4 \end{bmatrix}$

$\operatorname{tr}\begin{bmatrix} 2 & a \\ a & 4 \end{bmatrix} = 6 > 0$, $\det\begin{bmatrix} 2 & a \\ a & 4 \end{bmatrix} = 8 - a^2$

So $\mathbb{H}$ will have positive eignvalues if $8 - a^2 > 0$

$\Rightarrow \boxed{|a| < 2\sqrt{2}}$ under this condition the point

$\left(\frac{4+2a}{8-a^2}, \frac{-4-a}{8-a^2}, 1\right)$ is the local minimum of f.

3 (15 points) Use the method of Lagrange multipliers to find the minimum and maximum values $3x - y - 2z$ attains on the sphere defined by $x^2 + y^2 + z^2 = 1$.

$f := 3x - y - 2z, \quad \phi := x^2 + y^2 + z^2 - 1 = 0$

$F := 3x - y - 2z + \lambda(x^2 + y^2 + z^2 - 1)$

$$\left.\begin{aligned} F_x &= 3 + 2\lambda x = 0 \\ F_y &= -1 + 2\lambda y = 0 \\ F_z &= -2 + 2\lambda z = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} & x \neq 0,\ y \neq 0,\ z \neq 0 \ \& \\ & \lambda = -\frac{3}{2x} = \frac{1}{2y} = \frac{1}{z} \end{aligned}$$

$\Rightarrow \quad z = -\frac{2x}{3}, \quad y = -\frac{x}{3}$

$\phi = 0 \quad \Rightarrow \quad x^2 + \frac{x^2}{9} + \frac{4x^2}{9} = 1$

$\Rightarrow \quad \frac{14}{9} x^2 = 1 \quad \Rightarrow \quad \boxed{x = \pm \frac{3}{\sqrt{14}}}$

$\Rightarrow \quad \boxed{y = \mp \frac{1}{\sqrt{14}}} \quad \& \quad \boxed{z = \mp \frac{2}{\sqrt{14}}}$

$\exists$ 2 extremum points: $\pm \vec{p}$, $\vec{p} := \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$

$f(\pm\vec{p}) = \pm\left(\frac{9}{\sqrt{14}} + \frac{1}{\sqrt{14}} + \frac{4}{\sqrt{14}}\right) = \pm\sqrt{14}$

So max & min values are respectively $\sqrt{14}$ and $-\sqrt{14}$.

4 (10 points) Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{F}(x, y) = [e^x(y + \sin x^2) - y^3]\mathbf{i} + (e^x - \cos^2 y + x^3)\mathbf{j},$$

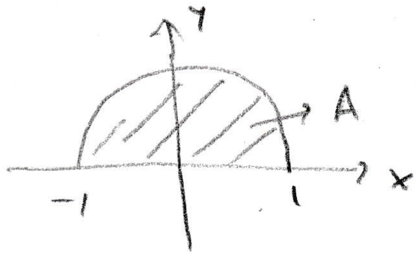
where x and y are Cartesian coordinates, and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointing along the positive x- and y-axes. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{x}$, where C is the counterclockwise-oriented boundary of the half disc defined by $\{\,(x, y) \in \mathbb{R}^2 \mid y \geq 0 \;\&\; x^2 + y^2 \leq 1\,\}$, and $\mathbf{x} := x\,\mathbf{i} + y\,\mathbf{j}$.

$$\oint_C \vec{F}\cdot d\vec{x} = \iint_A (\partial_1 F_2 - \partial_2 F_1)\, dx\, dy$$

$$F_1 = e^x(y + \sin x^2) - y^3, \quad F_2 = e^x - \cos^2 y + x^3$$

$$\partial_1 F_2 = \frac{\partial}{\partial x} F_2 = e^x + 3x^2$$

$$\partial_2 F_1 = \frac{\partial}{\partial y} F_1 = e^x - 3y^2$$



$$\Rightarrow \quad \partial_1 F_2 - \partial_2 F_1 = 3(x^2 + y^2)$$

$$\Rightarrow \quad \oint_C \vec{F}\cdot d\vec{x} = \iint_A 3(x^2+y^2)\, dx\, dy$$

$$= \int_0^1 \int_0^\pi 3r^2\, r\, d\theta\, dr$$ ← in polar coordinates

$$= \int_0^1 3r^3\, dr \int_0^\pi d\theta$$

$$= \left[3\left(\frac{r^4}{4}\right)\Big|_0^1\right](\pi)$$

$$= \frac{3\pi}{4}$$