Suggested Exercise Problems 2

Suppose m is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to m a subspace of $\mathcal{P}(\mathbf{F})$?

Prove that $\mathbf{F}^\infty$ is infinite dimensional.

Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n.

Suppose that V is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Suppose that $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $(p_0, p_1, \ldots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)& \\ &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Suppose V is finite dimensional. Prove that if $U_1, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$