Suggested Exercise Problems 5

Prove or give a counterexample: if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension j for each $j = 1, \dots, \dim V$.

Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.