Math 320: Quiz 2

20:00-20:50, Nov. 03, 2020

Problem 1 (4 pts, 10+3 minutes) Let $\mathbb{F}^\infty$ be the vector space of all sequences with terms belonging to $\mathbb{F}$, and for each $n \in \mathbb{Z}^+$ use $s^{(n)}$ to denote the sequence all of whose terms vanish except the n-th term which is equal to n, i.e.,

$$s^{(n)} := (s_1^{(n)}, s_2^{(n)}, \cdots, s_m^{(n)}, s_{m+1}^{(n)}, \cdots),$$

where for all $m \in \mathbb{Z}^+$,

$$s_m^{(n)} := \begin{cases} 0 & \text{for} \quad m \neq n, \\ n & \text{for} \quad m = n. \end{cases}$$

Let S be the set consisting of the sequences $s^{(n)}$. Is S a spanning subset of $\mathbb{F}^\infty$? Why?

For all $\sigma \in \mathrm{Span}(S)$, $\exists \ell \in \mathbb{Z}^+$, $\exists \alpha_1, \dots \alpha_\ell \in \mathbb{F}$, $\exists n_1, n_2, \dots, n_\ell \in \mathbb{Z}^+$,

$$\sigma = \alpha_1 s^{(n_1)} + \cdots + \alpha_\ell s^{(n_\ell)}$$

The m-th term of σ is

$$\sigma_m = \alpha_1 s_m^{(n_1)} + \cdots + \alpha_\ell s_m^{(n_\ell)}$$

Let $N := n_1 + \cdots + n_\ell + 1$ so that $\forall j \in \{1, \dots, n_\ell\}$, $N > n_\ell$

$\Rightarrow \forall m \in \mathbb{Z}^+$, $m \geq N \Rightarrow \sigma_m = 0$.

This show that σ has finitely many nonzero terms. $\Rightarrow$ Span(S) contain sequences with finitely many nonzero terms. But $\mathbb{F}^\infty$ includes sequences with nonzero terms, e.g., $(1, 1, 1, \cdots, 1, 1, \cdots)$. These do not belong to $S \Rightarrow S$ is not a spanning subset.

Problem 2 (10 pts, 17+3 minutes) Let V be the vector space of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree at most 11, i.e., $\exists \alpha_0, \alpha_1, \cdots, \alpha_{11} \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$p(x) := \sum_{j=0}^{11} \alpha_j x^j,$$

and W be the subspace of V consisting of even polynomials $q : \mathbb{R} \to \mathbb{R}$ that satisfy $q(1) = 0$. Find a basis for W. Justify your response.

Note: You may use the fact the set of monomials $p_j : \mathbb{R} \to \mathbb{R}$, which satisfy

$$\forall j \in \mathbb{N},\ \forall x \in \mathbb{R},\ \ p_j(x) := \begin{cases} 1 & \text{for} \quad j = 0, \\ x^j & \text{for} \quad j \geq 1, \end{cases}$$

is linearly independent.

$\forall q \in W$, degree$(q) \leq 11$ & q is even.
$\Rightarrow$ degree $(q) \leq 10$

$q(1) = 0 \Leftrightarrow r \in V$, $q(x) = (x-1)\, r(x)$

$q(-x) = q(x) \Leftrightarrow$ $q(x) = (-x-1)\, r(-x)$

$\Rightarrow$ $q(x) = -(x+1)\, r(-x)$

$\Rightarrow$ $q(1) = 0 \Rightarrow r(-1) = 0 \Rightarrow$ $\exists s \in V$,
$r(x) = (x+1)\, s(x)$

$\Rightarrow$ $q(x) = (x-1)(x+1)\, s(x) = (x^2-1)\, s(x)$ 1

q is even $\Rightarrow$ $s(x)$ must be even

degree $q \leq 10 \Leftrightarrow$ degree $s \leq 8$

$\exists\, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, $s(x) = \sum_{j=0}^{4} \alpha_j x^{2j}$ 2

let $\forall j \in \{0,1,2,3,4\}$, $q_j := (P_2 - 1) P_{2j}$ 3 $\Rightarrow$

(1-3) $\Rightarrow$ $q = (P_2 - 1) \sum_{j=0}^{4} \alpha_j P_{2j} = \sum_{j=0}^{4} \alpha_j q_j$

let $\mathcal{B} := (q_0, q_1, q_2, q_3, q_4) \Rightarrow q \in \text{Span}(\mathcal{B})$

$\Rightarrow W \subseteq \text{Span}(\mathcal{B})$ $\quad q_j \in W \Rightarrow \text{Span}(\mathcal{B}) \subseteq W$

$\Rightarrow W = \text{Span}(\mathcal{B})$.

$\forall \beta_0, \dots, \beta_4 \in \mathbb{R}$, $\sum_{j=0}^{4} \beta_j q_j = 0 \Rightarrow \sum_{j=0}^{4} \beta_j (P_2 - 1) P_{2j} = 0$

$\Rightarrow (P_2 - 1) \sum_{j=0}^{4} \beta_j P_{2j} = 0$, $P_2 - 1 \neq 0$

$\Rightarrow \sum_{j=0}^{4} \beta_j P_{2j} = 0$ $\quad \{P_j \mid j \in \mathbb{N}\}$ is linearly indep $\Rightarrow$

$\Rightarrow \beta_j = 0,\ \forall j \in \{0, \dots, 4\} \Leftrightarrow \mathcal{B}$ is linearly independent
$\Rightarrow \mathcal{B}$ is a basis of W.

Problem 3 (6 pts, 7+3 minutes) Let U, V, and W be vector spaces over $\mathbb{F}$, and $L \in \mathcal{L}(U, V)$ and $K \in \mathcal{L}(V, W)$. Show that that $K \circ L \in \mathcal{L}(U, W)$.

$\mathrm{dom}(L) = U$, $\mathrm{dom}(K) = V$

$\Rightarrow \forall u \in U, \exists v \in V, v = L(u)$

$\Downarrow$

$\exists w \in W, w = K(v)$ $\Rightarrow$ $w = K(L(u)) = (K \circ L)(u)$

$\Rightarrow u \in \mathrm{dom}(K \circ L) \Rightarrow U \subseteq \mathrm{dom}(K \circ L)$

But by definiti $\mathrm{dom}(K \circ L) \subseteq U \Rightarrow \boxed{\mathrm{dom}(K \circ L) = U}$

(1)

In particular $\mathrm{dom}(K \circ L)$ is a subspace of U.

$\forall u_1, u_2 \in U$, $\forall \alpha_1, \alpha_2 \in \mathbb{F}$,

$$\boxed{(K \circ L)(\alpha_1 u_1 + \alpha_2 u_2)} = K(L(\alpha_1 u_1 + \alpha_2 u_2))$$
$$= K(\alpha_1 L(u_1) + \alpha_2 L(u_2))$$
$$= \alpha_1 K(L(u_1)) + \alpha_2 K(L(u_2))$$
$$\boxed{= \alpha_1 (K \circ L)(u_1) + \alpha_2 (K \circ L)(u_2)}$$

(2)

(1) & (2) $\Rightarrow K \circ L \in \mathcal{L}(U, W)$.