Math 320: Quiz 3

20:00-20:50, Nov. 17, 2020

Problem 1 (6 pts, 10+3 minutes) Let

$$\boldsymbol{\sigma}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{\sigma}_4 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $B := (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_4)$ is a basis of $\mathbb{C}^{2\times 2}$. Find the matrix representation of $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ in this basis.

$$\alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3 + \alpha_4 \sigma_4 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_3 + \alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\begin{cases} \alpha_3 + \alpha_4 = 0 \\ -\alpha_3 + \alpha_4 = 3 \end{cases} \Rightarrow \begin{cases} \alpha_4 = \frac{3}{2} \\ \alpha_3 = -\frac{3}{2} \end{cases}$$

$$\begin{cases} \alpha_1 + i\alpha_2 = 2 \\ \alpha_1 - i\alpha_2 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{3}{2} \\ \alpha_2 = -\frac{i}{2} \end{cases}$$

$$\Rightarrow m_B\left(\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} \\ -\frac{i}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Problem 2 (6 pts, 10+3 minutes) Let $T \in \mathcal{L}(\mathbb{C}^{2\times 2})$ be given by $T\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \alpha+\delta & \beta-\gamma \\ \gamma-\beta & 0 \end{bmatrix}$. Find the matrix representation of T in the basis $B := (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_4)$ with

$$\boldsymbol{\sigma}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \boldsymbol{\sigma}_4 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T\sigma_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{M}_B(T\sigma_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\sigma_2 = \frac{1}{2}\begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix} = \sigma_2 \Rightarrow \mathcal{M}_B(T\sigma_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\sigma_3 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{M}_B(T\sigma_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\sigma_4 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\sigma_3 + \frac{1}{2}\sigma_4$$

$$\Rightarrow \mathcal{M}_B(T\sigma_4) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \Rightarrow \mathcal{M}_{B,B}(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Problem 3 (8 pts, 17+3 minutes) Let $\mathbb{R}[x]$ be the vector space of real polynomias, $p : \mathbb{R} \to \mathbb{R}$, and U be the subspace of $\mathbb{R}[x]$ consisting of odd polynomials, i.e.,

$$U := \{\, p \in \mathbb{R}[x] \mid \forall x \in \mathbb{R},\ p(-x) = -p(x) \,\}.$$

Show that U is isomorphic to $\mathbb{R}[x]$.

$\forall p \in \mathbb{R}[x]$, $\exists n \in \mathbb{N}$, $\exists \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\alpha_n \neq 0$ and

$\forall x \in \mathbb{R}$, $\quad p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = \sum_{j=0}^{n} \alpha_j x^j$

Define $T : \mathbb{R}[x] \to U$ by

$$(T(p))(x) := \sum_{j=0}^{n} \alpha_j x^{2j+1}$$

Clearly $(T(p))(-x) = \sum_{j=0}^{n} \alpha_j (-x)^{2j+1} = \sum_{j=0}^{n} \underbrace{(-1)^{2j+1}}_{-1} \alpha_j x^{2j+1}$

$$= -(T(p))(x)$$

$\Rightarrow \quad T(p) \in U$

$\forall p, \tilde{p} \in \mathbb{R}[x]$

let $n, \tilde{n} \in \mathbb{N}$, $\alpha_0, \dots, \alpha_n, \tilde{\alpha}_0, \dots, \tilde{\alpha}_{\tilde{n}} \in \mathbb{R}$, $\alpha_{\tilde{n}} \neq 0$ & $\tilde{\alpha}_{\tilde{n}} \neq 0$

$\forall x \in \mathbb{R}$, $\quad p(x) = \sum_{j=0}^{n} \alpha_j x^j$, $\quad q(x) = \sum_{k=0}^{\tilde{n}} \tilde{\alpha}_k x^k$

① If $p = q \Rightarrow n = \tilde{n}$ & $\forall j \in \{1, \dots, n\}$, $\alpha_j = \tilde{\alpha}_j$.

$\Rightarrow (T(p))(x) = \sum_{j=0}^{n} \alpha_j x^{3j+1} = \sum_{j=0}^{\tilde{n}} \hat{\alpha}_j x^{2j+1} = (T(\hat{p}))(x)$

$\Rightarrow \quad T(p) = T(\tilde{p}) \quad \Rightarrow T$ is a function.

② If $T(p) = T(\tilde{p}) \Rightarrow \sum_{j=0}^{n} \alpha_j x^{2j+1} = \sum_{k=0}^{\tilde{n}} \tilde{\alpha}_k x^{2k+1}$

$\Rightarrow \tilde{n} = n$ & $\forall j \in \{1, \dots, n\}$, $\alpha_j = \tilde{\alpha}_j$. $\Rightarrow \quad p = \hat{p}$

$\Rightarrow T$ is 1-to-1.

③ $\forall q \in U$, $\exists m \in \mathbb{N}$, $\exists \alpha_0, \dots, \alpha_m \in \mathbb{R}$,

$\forall x \in \mathbb{R}$, $q(x) = \sum_{j=0}^{m} \alpha_j x^{2j+1}$. Let $p \in \mathbb{R}[x]$ be defined by

$\forall x \in \mathbb{R}, \quad p(x) := \sum_{j=0}^{m} \alpha_j x^j \quad \Rightarrow$

$(T(p))(x) = \sum_{j=0}^{m} \alpha_j x^{2j+1} = q(x) \Rightarrow T(p) = q$

$\Rightarrow T$ is onto.

④ Because we define Tp for every $p \in \mathbb{R}[x]$,
$\mathrm{dom}(T) = \mathbb{R}[x]$

Let $p, \tilde{p} \in \mathbb{R}[x]$, $n := \deg(p)$, $\tilde{n} := \deg(\tilde{p})$,
$\exists \alpha_0, \dots, \alpha_n, \tilde{\alpha}_0, \dots, \tilde{\alpha}_{\tilde{n}} \in \mathbb{R}$, $\forall x \in \mathbb{R}$

$p(x) = \sum_{j=0}^{n} \alpha_j x^j \quad \& \quad \tilde{p}(x) = \sum_{k=0}^{\tilde{n}} \tilde{\alpha}_k x^k$

$\forall \beta, \tilde{\beta} \in \mathbb{R}$

$$(\beta p + \tilde{\beta}\tilde{p})(x) = \beta \sum_{j=0}^{n} \alpha_j x^j + \tilde{\beta} \sum_{k=0}^{\tilde{n}} \tilde{\alpha}_k x^k$$

$$= \sum_{j=0}^{n} \beta \alpha_j x^j + \sum_{k=0}^{\tilde{n}} \tilde{\beta} \tilde{\alpha}_k x^k$$

$$\Rightarrow (T(\beta p + \tilde{\beta}\tilde{p}))(x) = \sum_{j=0}^{n} \beta \alpha_j x^{2j+1} + \sum_{k=0}^{\tilde{n}} \tilde{\beta} \tilde{\alpha}_k x^{2k+1}$$

$$= \beta \sum_{j=0}^{n} \alpha_j x^{2j+1} + \tilde{\beta} \sum_{k=0}^{\tilde{n}} \tilde{\alpha}_k x^{2k+1}$$

$$= \beta (T(p))(x) + \tilde{\beta} (T(\tilde{p}))(x)$$

$$= (\beta T(p) + \tilde{\beta} T(\tilde{p}))(x)$$

$\Rightarrow T(\beta p + \tilde{\beta}\tilde{p}) = \beta T(p) + \tilde{\beta} T(\tilde{p}) \Rightarrow T$ is linear.

①-④ T is a linear bijection $\Rightarrow T$ is invertible $\Rightarrow \mathbb{R}[x]$ is isom. to U
$\Rightarrow U$ is isom. to $\mathbb{R}[x]$.