Math 320: Quiz 4

20:00-20:50, Dec. 01, 2020

Problem 1 (8 pts, 15+3 minutes) Let V be a finite-dimensional vector space over $\mathbb{F}$, I be the identity operator acting in V, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F} \setminus \{0\}$ such that

$$V = \text{null}(T - \lambda I) \oplus \text{null}(T + \lambda I).$$

Show that there is a basis of B in which T is represented by a diagonal matrix.

Let $U_+ := \text{null}(T - \lambda I)$ & $U_- := \text{null}(T + \lambda I)$

Let $n_+ \overset{①}{:=} \dim U_+$ & $n_- \overset{②}{:=} \dim U_-$. Then

$\dim V \overset{③}{=} n_+ + n_-$. This follows from $U_+ \cap U_- = \{0\}$ &

$\dim V = \dim(U + V) = \dim U + \dim V - \dim U \cap V$.

(1) & (2) $\Rightarrow \exists\, b_1, b_2, \dots, b_{n_+}, c_1, c_2, \dots, c_{n_-} \in V$,

$B := (b_1, \dots, b_{n_+})$ is a basis of U_+ and

$C := (c_1, \dots, c_{n_-})$ " " " " U_-.

Let $D := (b_1, \dots, b_{n_+}, c_1, \dots, c_{n_-})$. Then $\forall\, \alpha_1, \dots, \alpha_{n_+}, \beta_1, \dots, \beta_{n_-} \in \mathbb{F}$,

$\alpha_1 b_1 + \dots + \alpha_{n_+} b_{n_+} + \beta_1 c_1 + \dots + \beta_{n_-} c_{n_-} \overset{④}{=} 0 \Rightarrow$

$(T - \lambda I)\left(\sum_{i=1}^{n_+} \alpha_i b_i + \sum_{j=1}^{n_-} \beta_j b_j\right) = 0 \Rightarrow \sum_{i=1}^{n_+} \alpha_i (T - \lambda I) b_i + \sum_{j=1}^{n_-} \beta_j (T - \lambda I) c_j \overset{⑤}{=} 0$

$b_i \in U_+ \Rightarrow (T - \lambda I) b_i = 0$

$c_i \in U_- \Rightarrow (T + \lambda I) c_j = 0 \Rightarrow (T - \lambda I) c_j = (T + \lambda I - 2\lambda I) c_j = -2\lambda c_j$

$\overset{⑤}{\Rightarrow} \underset{\neq 0}{-2\lambda} \sum_{j=1}^{n_-} \beta_j c_j = 0 \Rightarrow \sum_{j=1}^{n_-} \beta_j c_j = 0 \Rightarrow \beta_j = 0\ \forall j \in \{1 \cdots n_-\}$ because C is linearly indep.

$\overset{④}{\Rightarrow} \sum_{i=1}^{n_+} \alpha_i b_i = 0 \Rightarrow \alpha_i = 0\ \forall i \in \{1, \dots n_+\}$ because B is linearly ind.

$\Rightarrow D$ is linearly indep. $|D| = n_+ + n_- = n = \dim V \Rightarrow D$ is a basis of V.

$T b_j = \lambda b_j$ & $T c_j = (T + \lambda I - \lambda I) c_j = -\lambda c_j$

$\Rightarrow D$ is a basis consisting of eigenvectors of T.

$\Rightarrow \mathcal{M}_{D,D}(T)$ is diagonal. $\mathcal{M}_{D,D}(T) = \begin{bmatrix} \lambda & & & & 0 \\ & \ddots & & & \\ & & \lambda & & \\ & & & -\lambda & \\ 0 & & & & \ddots\ -\lambda \end{bmatrix}$.

Problem 2 (6 pts, 12+3 minutes) Let V be a three-dimensional vector space, $T \in \mathcal{L}(V)$, and U and W are two-dimensional invariant subspaces for T such that $V = U + W$. Show that there is a basis of V in which T is represented by an upper-diagonal matrix.

$$3 = \dim(V) = \dim(U+W) = \underbrace{\dim(U)}_{=2} + \underbrace{\dim(W)}_{=2} - \dim(U \cap W)$$

$\Rightarrow \dim(U \cap W) = 1$

$\forall v \in U \cap W$, $v \in U$ and $v \in W$

$\Rightarrow Tv \in U$ and $Tv \in W \Rightarrow Tv \in U \cap W$

$\Rightarrow U \cap W$ is an invariant subspace for T.

Let $\mathcal{B}_0$ be a basis of $U \cap W$. Because $\dim(U \cap W) = 1$, $\exists\, b_1 \in V$, $\mathcal{B}_0 = (b_1) \Rightarrow b_1 \in U \setminus \{0\} \Rightarrow \mathcal{B}_0$ is a linearly independent list in $U \Rightarrow$ we can extend it to a basis $\mathcal{B}_1$ of U. Because $\dim U = 2$, $\exists\, b_2 \in U$, $\mathcal{B}_1 = (b_1, b_2)$. $\mathcal{B}_1$ is a linearly independent list in V.

$\Rightarrow$ we can extend it to a basis $\mathcal{B}$ of V

$\dim V = 3 \Rightarrow \exists\, b_3 \in V$, $\mathcal{B} = (b_1, b_2, b_3)$.

Now, $Tb_1 \in \operatorname{Span}(b_1) = U \cap W$ because $U \cap W$ is an invariant subspace for T

$Tb_2 \in \operatorname{Span}(b_1, b_2) = U$ because U is an invariant subspace for T

$Tb_3 \in \operatorname{Span}(b_1, b_2, b_3) = V$

$\Rightarrow \forall j \in \{1,2,3\}$, $Tb_j \in \operatorname{Span}(b_1, \dots, b_j) \Rightarrow T$ has an upper-triangular matrix representation in the basis $\mathcal{B}$.

$\blacksquare$

Problem 3 (6 pts, 10+3 minutes) Let V be the vector space of polynomials $p : [0,1] \to \mathbb{C}$ of degree at most 1,

$$\forall j \in \{0,1\},\ \forall x \in [0,1],\ p_j(x) := \begin{cases} 1 & \text{for } \ j=0, \\ x & \text{for } \ j=1, \end{cases}$$

and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be inner product defined by

$$\forall p, q \in V,\ \langle p, q \rangle := \int_0^1 p(x)\,\overline{q(x)}\,dx.$$

Construct an orthonormal basis of the complex inner-product space $(V, \langle \cdot, \cdot \rangle)$ by performing the Gram-Schmidt orthogonalization procedure on (p_0, p_1).

$q_0 := \frac{1}{\|p_0\|} p_0 \qquad \|p_0\|^2 = \int_0^1 dx = 1 \implies \boxed{q_0 = p_0}$

$r_1 := p_1 - \langle p_1, q_0 \rangle q_0$

$\langle p_1, q_0 \rangle = \langle p_1, p_0 \rangle = \int_0^1 p_1(x)\,\overline{p_0(x)}\,dx = \int_0^1 x\,dx = \frac{1}{2}$

$\implies r_1(x) = p_1(x) - \frac{1}{2} q_0(x) = x - \frac{1}{2}$

$q_1 := \frac{1}{\|r_1\|} r_1$

$\|r_1\|^2 = \langle r_1, r_1 \rangle = \int_0^1 |r_1(x)|^2 dx = \int_0^1 (x - \frac{1}{2})^2 dx$

$= \int_0^1 (x^2 - x + \frac{1}{4})^2 dx = (\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}) \Big|_0^1$

$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12} \implies \|r_1\| = \frac{1}{2\sqrt{3}}$

$\implies q_1(x) = 2\sqrt{3}(x - \frac{1}{2}) \implies \boxed{q_1 = -\sqrt{3}\, p_0 + 2\sqrt{3}\, p_1}$

$\mathcal{E} = \{q_0, q_1\}$ is an orthonormal basis for V.