Math 320: Quiz 1, Part 1

20:00-20:50, Oct. 20, 2020

Problem 1 (6 pts, 10+3 minutes) Let V be the real vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$, and U be the subset of V consisting of functions $g : \mathbb{R} \to \mathbb{R}$ such that there is some $a \in \mathbb{R}$, $g(x) = 0$ for all $x \geq a$, i.e.,

$$U := \Big\{ g \in V \;\Big|\; \exists a \in \mathbb{R},\ \forall x \in [a, \infty),\ g(x) = 0 \Big\}.$$

Show that U is a subspace of V.

Problem 2 (10+3 minutes) Let V be a vector space over $\mathbb{F}$, A and B be subsets of V that are NOT necessarily subspaces, and

$$A + B := \Big\{ v \in V \;\Big|\; \exists a \in A,\ \exists b \in B,\ v = a + b \Big\}.$$

2.a (4 pts) Is $\mathrm{Span}(A + B) \subseteq \mathrm{Span}(A) + \mathrm{Span}(B)$? Why?

2.b (4 pts) Is $\mathrm{Span}(A) + \mathrm{Span}(B) \subseteq \mathrm{Span}(A + B)$? Why?

Problem 3 (6 pts, 17+3 minutes) Consider the complex vector space obtained by endowing $\mathbb{C}^2$ with componentwise addition and scalar multiplication, $\mathbf{0} := (0, 0)$, $\mathbf{a} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$, $\alpha_1, \alpha_2 \in \mathbb{C}$ be the components of $\mathbf{a}$, so that $\mathbf{a} = (\alpha_1, \alpha_2)$, and $\overline{\mathbf{a}} := (\overline{\alpha_1}, \overline{\alpha_2})$ be the complex-conjugate of $\mathbf{a}$. Here, for all $j \in \{1, 2\}$, $\overline{\alpha_j}$ stands for the complex-conjugate of α_j. Let $U_1 := \mathrm{Span}(\{\mathbf{a}\})$ and $U_2 := \mathrm{Span}(\{\overline{\mathbf{a}}\})$. Find a necessary and sufficient condition on α_1 and α_2 under which $\mathbb{C}^2 = U_1 \oplus U_2$. Justify your response.

Note: In your response, you may use your knowledge of the solutions of systems of linear algebraic equations that you have treated in Math 107.