MATH 320 - LINEAR ALGEBRA
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SPRING 2005, Quiz #1-Solutions by Turker Ozsari

You may not get credit unless you show all your work and write neatly!
You have 30 minutes.

(1) V is a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other. (10 Points)

Proof.

($\Rightarrow$) Let U and W be two subspaces of V such that $U \bigcup W$ is a subspace of V. Assume $U/W \neq \emptyset$ and $W/U \neq \emptyset$. Let $u \in U/W$ and $w \in W/U$ be two elements. Then, $u + w \in U \bigcup W$, which implies either $u + w \in U$ or $u + w \in W$. In the first case, there is an $x \in U$ such that $x = u + w$. Then $w = x - u \in U$, contradiction. In the latter case, the same procedure gives $u \in W$, contradiction. Hence, we cannot have both $U/W \neq \emptyset$ and $W/U \neq \emptyset$. Thus either $U/W = \emptyset$ or $W/U = \emptyset$, that is, either $U \subseteq W$ or $W \subseteq U$.

($\Leftarrow$) Let U, W be subspaces of V such that $U \subseteq W$. Then, $U \bigcup W = U$ which is a subspace of V by assumption.

(2) Prove that the vector space V of complex polynomials over $\mathbb{C}$ (complex numbers) is infinite-dimensional. (10 Points)

Proof.

Assume V is finite dimensional. Then, there is a spanning list of polynomials $(p_1, p_2, ..., p_n)$ in V. Let $d(p_i)$=degree(p_i) and

$$m = \max_{1 \leq i \leq n} d(p_i)$$

Then, any linear combination of $p_i's$ has degree less than or equal to m. Thus, one cannot get a polynomial with degree greater than m as a linear combination of $p_1, ..., p_n$, which contradicts the fact that $(p_1, p_2, ..., p_n)$ spans V. Hence, our assumption is false, that is, V is infinite-dimensional.