MATH 320 - LINEAR ALGEBRA
Instructor: Ali Mostafazadeh
SPRING 2005, Quiz #3

You may not get credit unless you show all your work and write neatly!
You have 55 minutes.

(1) Prove or disprove: if $U \leq V$ and invariant under every (linear) operator on V, then $U = \{0\}$ or $U = V$. (10 Points)
Proof: If not, $0 < m :=$ dim$(U) < n :=$ dim$V$. Let $(v_1, ..., v_m)$ be a basis of U and extend it to a basis of V, $(v_1, ..., v_n)$. Define, $Tv_1 := v_n, Tv_n := v_1, Tv_i := v_i, 1 < i < n$. Then, we must have $Tv_1 = v_n \in U$, contradiction.■
(2) $p \in P(\mathbb{C})$, deg$(p) = m$. Prove that p has m distinct roots iff p and its derivative p' have no roots in common. (10 points)
Proof: ($\Rightarrow$) Let $p(z) = c\prod_{i=1}^{m}(z-\lambda_i), \lambda_i \neq \lambda_j$ for $i \neq j$. Then, $p'(z) = c\sum_{j=1}^{m}\prod_{i\neq j,i=1}^{m}(z-\lambda_i)$. Then, $p'(\lambda_k) = c\prod_{i\neq k,i=1}^{m}(\lambda_k - \lambda_i) \neq 0$, since $\lambda_i \neq \lambda_k$ for $i \neq k$.
($\Leftarrow$) Suppose $\exists\lambda \ni p(z) = c(z-\lambda)^2 q(z), q \in P(\mathbb{C})$. Then, $p'(z) = 2c(z-\lambda)q(z) + c(z-\lambda)^2 q'(z)$. Hence, $p'(\lambda) = 0$, i.e., p and p' have a common root.■