MATH 320: QUIZ-4, Solutions

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Time: 35 min

April 18, 2005

1. Suppose T be a linear operator on a finite dimensional vector space V such that for every $v \in V$, $Tv = T(Tv)$. Prove that $V = \text{null } T \bigoplus \text{range } T$. (10 points)

Proof. Let $v \in V$. Then $v = T(v) + v - T(v)$, but $T(v - T(v)) = T(v) - T^2(v) = T(v) - T(v) = 0$, hence $v - T(v) \in \text{null } T$. Hence, $V = \text{null } T + \text{range } T$. Now, let $v \in \text{null } T \cap \text{range } T$. Then, $T(v) = 0$ and $\exists u \in V \ni T(u) = v$, but then $v = T(u) = T(T(u)) = T(v) = 0$. Hence, $\text{null } T \cap \text{range } T = \{0\}$. $**$

2. Let T be a linear operator on a **complex** inner product space V such that $(Tx, x) = 0$ for every $x \in V$. Prove T is the zero operator. (10 points)

Proof. Let $u, v \in V$. Then, $4(Tu, v) = (T(u + v), u + v) - (T(u - v), u - v) + i(T(u + iv), u + iv) - i(T(u - iv), u - iv)$, but since $(Tx, x) = 0$, we have $(Tu, v) = 0$. Take, $v = Tu$, we have $||Tu||^2 = 0$, hence $Tu = 0$ for all $u \in V$. $**$