Math 320: Quiz # 2

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	Solutions

- You have 50 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- No question are answered during this quiz.

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1) Let V and W be vector spaces over $\mathbb{F}$ and $L : V \to W$ be a one-to-one linear map with range R. The inverse of L is a function $L^{-1} : W \to V$ with domain R such that for all $w \in R$, $L^{-1}w = v$ if $w = L(v)$.

1.a (2 points) What is the range of L^{-1}.

$$Ran(L^{-1}) = \{v \in V : L^{-1}(w) = v,\ w \in R\} = \{v \in V : L(v) = w,\ w \in R\}$$
$$= Dom(L) \leq V$$

1.b (8 points) Show that L^{-1} is a linear map.

(Note that L^{-1} is well-defined function, since L is 1-1)

L^{-1} is a linear map, because

(i) $Dom(L^{-1}) = Ran(L)$ is a subspace of W.

(ii) $w_1, w_2 \in R \Rightarrow \exists v_1, v_2 \in V$ s.t. $L^{-1}(w_1) = v_1$, $L^{-1}(w_2) = v_2$ satisfying $L(v_1) = w_1$, $L(v_2) = w_2$.

Since L is linear $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$.

$$\Rightarrow L^{-1}(w_1 + w_2) = \underbrace{v_1}_{L^{-1}(w_1)} + \underbrace{v_2}_{L^{-1}(w_2)} \Rightarrow L^{-1}(w_1) + L^{-1}(w_2) = L^{-1}(w_1 + w_2).$$

(iii) Let $\alpha \in \mathbb{F}$, $w \in R$. Then $\exists v \in V$ s.t. $w = L(v)$.

$\alpha w = \alpha L(v) = L(\alpha v)$. This implies that $L^{-1}(w) = v$ and $L^{-1}(\alpha w) = \alpha v$. Thus, $L^{-1}(\alpha w) = \alpha L^{-1}(w)$.

2) Let V and W be vector spaces over $\mathbb{F}$, V be finite-dimensional, and $L \in \mathcal{L}(V, W)$. Prove the following statements.

2.a (2 points) $\mathrm{Nul}(L)$ is finite-dimensional.

Null(L) is a subspace of V. V is finite dimensional, so is Null(L).

2.b (5 points) $\mathrm{Ran}(L)$ is finite-dimensional.

$w \in Ran(L) \Rightarrow \exists v \in V$ s.t. $w = L(v)$. Let $(v_1, \dots v_n)$ be a basis of V

Then $\exists \alpha_1 \dots \alpha_n \in \mathbb{F}$, $v = \alpha_1 v_1 + \dots + \alpha_n v_n$.

$w = L(v) = L(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 L(v_1) + \dots + \alpha_n L(v_n)$

So $Ran(L)$ can be spanned by $(L(v_1), \dots, L(v_n))$. Thus, it is finite dimensional.

2.c (13 points) $\dim(\mathrm{Nul}(L)) + \dim(\mathrm{Ran}(L)) = \dim(V)$.

Suppose $(v_1, \dots v_m)$ is a basis of Null(L). (If it is empty, it is basis of $\{0\}$)

Let $\dim V = n$. If $m = n$, $\underbrace{\dim V}_{n} = \underbrace{\dim(Ran(L))}_{0} + \underbrace{\dim Null(L)}_{m}$, since $L(v) = 0 \ \forall v \in V$.

If $m < n$, we can extend $(v_1 \dots v_m)$ to a basis $(v_1 \dots v_m, u_1 \dots u_{n-m})$ of V.

Claim: $(L(u_1), \dots, L(u_{n-m}))$ is a basis of $Ran(L)$.

Proof of claim: (i) $w \in Ran(L) \Rightarrow \exists v \in V$ s.t. $w = L(v)$

v can be written as a linear combination of basis elts of V:

$v = a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_{n-m} u_{n-m}$, $a_1 \dots a_m, b_1, \dots b_{n-m} \in \mathbb{F}$.

$\Rightarrow w = L(v) = L(b_1 u_1 + \dots + b_{n-m} u_{n-m}) = b_1 L(u_1) + \dots + b_{n-m} L(u_{n-m})$

because $a_1 v_1 + \dots + a_m v_m \in$ Null(L).

So, $Ran(L)$ is spanned by $(L(u_1), \dots, L(u_{n-m}))$

(ii) Let $\alpha_1 \dots \alpha_{n-m} \in \mathbb{F}$.

$\alpha_1 L(u_1) + \dots + \alpha_{n-m} L(u_{n-m}) = 0 \Rightarrow L(\alpha_1 u_1 + \dots + \alpha_{n-m} u_{n-m}) = 0$

$\Rightarrow \alpha_1 u_1 + \dots + \alpha_{n-m} u_{n-m} \in$ Null(L)

$\Rightarrow \exists \beta_1, \dots \beta_m$ s.t. $\alpha_1 u_1 + \dots + \alpha_{n-m} u_{n-m} = \beta_1 v_1 + \dots + \beta_m v_m$

$\Rightarrow \alpha_1 u_1 + \dots + \alpha_{n-m} u_{n-m} - \beta_1 v_1 - \dots - \beta_m v_m = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots \alpha_{n-m} = \beta_1 = \dots = \beta_m = 0$

because $(u_1 \dots u_{n-m} v_1 \dots v_m)$ is a linearly ind list.

Hence, $(L(u_1) \dots L(u_{n-m}))$ is a linearly ind list and by (i) it is a basis of $Ran(L)$.

$\therefore \dim(Ran(L)) = n - m$, $\dim(Null(L)) = m$ $\dim V = n = \dim(Ran(L)) + \dim(Null(L))$

3 (10 points) Let V and W be vector spaces over $\mathbb{F}$ and $L \in \mathcal{L}(V, W)$. Show that if Nul(L) and Ran(L) are finite-dimensional, then V is also finite-dimensional.

Since Ran(L) is finite dimensional, there exists a basis $(w_1, \dots, w_m)$ of Ran(L). By the defn of Ran(L) there exist $v_1 \dots v_m \in V$ s.t. $w_1 = L(v_1), \dots, w_m = L(w_m)$

$$v \in V \Rightarrow L(v) = a_1 w_1 + \dots + a_m w_m \quad , a_1, \dots a_m \in \mathbb{F}$$
$$= a_1 L(v_1) + \dots + a_m L(v_m)$$
$$\Rightarrow L(v) = L(a_1 v_1 + \dots + a_m v_m)$$
$$\Rightarrow L(v - a_1 v_1 - \dots - a_m v_m) = 0$$
$$\Rightarrow v - a_1 v_1 - a_2 v_2 - \dots - a_m v_m \in Null(L).$$

It is given that Null(L) is also finite dimensional. Hence, $v - a_1 v_1 - a_2 v_2 - \dots - a_m v_m$ can be written as a linear combination of finite number of elts. in V or equal to 0. Then, v can be written as a linear combination of finite number of elts. in V. In other words, V is spanned by a finite list of elts.
Therefore, V is finite dimensional.