Math 320: Quiz # 3

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- No question are answered during this quiz.

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1) (10 points) Let V and W be inner-product spaces and $T \in \mathcal{L}(V, W)$. Show that $\mathrm{Nul}(T^*) = \mathrm{Ran}(T)^\perp$.

Let $w \in W$,

$w \in Null(T^*) \Leftrightarrow T^* w = 0$

$\Leftrightarrow \langle v, T^* w \rangle = 0, \ \forall v \in V$

$\Leftrightarrow \langle Tv, w \rangle = 0 \ \ \forall v \in V$

$\Leftrightarrow w \in Ran(T)^\perp$

2) (12 points) Let V be an inner-product space, $T \in \mathcal{L}(V)$, U be a subspace of V, and P_U be the orthogonal projection onto U. Show that U and $U^\perp$ are invariant subspaces of T if and only if $P_U T = T P_U$.

($\Rightarrow$) Suppose U and $U^\perp$ are invariant subspaces of T

i.e. $TU \subseteq U$, $TU^\perp \subseteq U^\perp$

Since $V = U \oplus U^\perp$, for any $v \in V$ $\exists u \in U$ and $\exists w \in U^\perp$ s.t.

$v = u + w$.

$$P_U T v = P_U T(u+w) = P_U(Tu + Tw) = Tu$$, because

by the assumption, $Tu \in U$ and $Tw \in U^\perp$.

$$T P_U v = T P_U (u+w) = Tu$$

So, $P_U T = T P_U$.

($\Leftarrow$) Suppose $P_U T = T P_U$.

Let $u \in U$, $P_U T u = Tu \Rightarrow Tu \in \operatorname{Ran} P_U = U$, i.e. $TU \subseteq U$.

Similiarly, $w \in U^\perp \Rightarrow P_U T w = 0 \Rightarrow Tw \in \operatorname{Null} P_U = U^\perp$

i.e. $TU^\perp \subseteq U^\perp$.

3 (10 points) Let V and W be inner-product spaces and $T \in \mathcal{L}(V, W)$. Show that T is one-to-one if and only if T^* is onto.

T is 1-1 $\Leftrightarrow \text{Nul}(T) = \{0\}$

$\Leftrightarrow \text{Ran}(T^*)^\perp = \{0\}$] (1) Take T to T^* in question $T^{**} = T$

$\Leftrightarrow \text{Ran}(T^*) = W$

$\Leftrightarrow T$ is onto.

4 (8 points) Let V be a complex inner-product space and $T \in \mathcal{L}(V)$ be a self-adjoint operator. Show that eigenvalues of T are real.

Let T be a self-adjoint operator, λ be an eigenvalue of T s.t. $Tv = \lambda v$, where v is a nonzero vector.

$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle$

$\langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$

Since $\langle Tv, v\rangle = \langle v, Tv\rangle$ and v is nonzero

$\lambda = \bar{\lambda}$. So, λ is real.