Solution

Math 450/550: Midterm Exam 1

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours.

Problem 1 Consider a classical particle moving on the real line so that its phase space $\mathscr{P} = \{(p,q) \mid p,q \in \mathbb{R}\} = \mathbb{R}^2$. Let $Q, P \in C^\infty(\mathscr{P})$ be the observables defined by $Q(q,p) := q$ and $P(q,p) := p$. Let ω_1 and ω_2 be pure states of this system which are respectively given by $(q_1,p_1) := (1,1)$ and $(q_2,p_2) := (1,-1)$, and $\omega := (2\omega_1 + 3\omega_2)/5$.

1.a (10 points) Find the expectation value of Q and P in the state ω.

$\rho_i \longleftrightarrow \omega_i$,
$$\rho_1(q,p) = \delta(q-q_1)\delta(p-p_1) = \delta(q-1)\delta(p-1)$$
$$\rho_2(q,p) = \delta(q-q_2)\delta(p-p_2) = \delta(q-1)\delta(p+1)$$

$$\langle Q|\omega\rangle = \frac{2}{5}\langle Q|\omega_1\rangle + \frac{3}{5}\langle Q,\omega_2\rangle$$
$$= \frac{2}{5}\underbrace{\int_{\mathbb{R}^2} \delta(q-1)\delta(p-1)\,q\,dq\,dp}_{1} + \frac{3}{5}\underbrace{\int_{\mathbb{R}^3} \delta(q-1)\delta(p+1)\,q\,dq\,dp}_{1}$$
$$= \frac{2}{5} + \frac{3}{5} = 1$$

$$\langle P|\omega\rangle = \frac{2}{5}\langle P|\omega_1\rangle + \frac{3}{5}\langle P|\omega_2\rangle$$
$$= \frac{2}{5}\underbrace{\int_{\mathbb{R}^2} \delta(q-1)\delta(p-1)\,p\,dq\,dp}_{1} + \frac{3}{5}\underbrace{\int_{\mathbb{R}^2} \delta(q-1)\delta(p+1)\,p\,dq\,dp}_{-1}$$
$$= \frac{2}{5} - \frac{3}{5}$$
$$= -\frac{1}{5}$$

1.b (10 points) Find the variance (standard deviation) of Q and P in the state ω.

$$\langle Q^2 | \omega \rangle = \frac{2}{5} \underbrace{\int_{\mathbb{R}^2} \delta(q-1)\,\delta(p-1)\, q^2 \, dq\, dp}_{1} + \frac{3}{5} \underbrace{\int_{\mathbb{R}^2} \delta(q-1)\,\delta(p+1)\, q^2 \, dq\, dp}_{1}$$

$$= 1$$

$$\Rightarrow \Delta_\omega Q = \sqrt{\langle Q^2 | \omega \rangle - \langle Q | \omega \rangle^2} = \sqrt{1-1} = 0$$

$$\langle P^2 | \omega \rangle = \frac{2}{5} \underbrace{\int_{\mathbb{R}^2} \delta(q-1)\,\delta(p-1)\, p^2 \, dq\, dp}_{1} + \frac{3}{5} \underbrace{\int_{\mathbb{R}^2} \delta(q-1)\,\delta(p+1)\, p^2 \, dq\, dp}_{1}$$

$$= 1$$

$$\Rightarrow \Delta_\omega P = \sqrt{\langle P^2 | \omega \rangle - \langle P | \omega \rangle^2} = \sqrt{1 - \left(-\frac{1}{5}\right)^2} = \sqrt{\frac{24}{25}}$$

$$= \frac{2\sqrt{6}}{5}$$

Problem 2 Let n be a positive integer, $(V_1, \langle\cdot,\cdot\rangle_1)$ and $(V_2, \langle\cdot,\cdot\rangle_2)$ be n-dimensional complex inner-product spaces, $\mathscr{E} := \{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of V_1, $\mathscr{F} := \{f_1, f_2, \cdots, f_n\}$ be an orthonormal basis of V_2, and $U : V \to W$ be a linear operator whose domain contains $\mathscr{E}_1$ and for all $j \in \{1, 2, \cdots, n\}$, $Ue_j = f_j$.

2.a (2 points) Show that the domain of U is V_1.

$\forall v \in V_1,\ \exists!\, \alpha_1, \dots, \alpha_n \in \mathbb{C},\ v = \sum_{j=1}^{n} \alpha_j e_j$

$e_j \in \mathrm{Dom}(U)$ & $\mathrm{Dom}(U)$ is a subspace $\Rightarrow v \in \mathrm{Dom}(U)$

$\Rightarrow V_1 \subseteq \mathrm{Dom}(U) \subseteq V_1 \Rightarrow \mathrm{Dom}(U) = V_1$.

2.b (18 points) Show that U is a unitary operator.

$$Uv = U\sum_{j=1}^{n} \alpha_j e_j = \sum_{j=1}^{n} \alpha_j Ue_j = \sum_{j=1}^{n} \alpha_j f_j$$

$\langle Uv, Uv\rangle_2 = \sum_{j=1}^{n} |\alpha_j|^2$ because $\mathcal{F}$ is orthonormal.

$v = \sum_{j=1}^{n} \alpha_j v_j \Rightarrow \langle v, v\rangle_1 = \sum_{j=1}^{n} |\alpha_j|^2$ because $\mathcal{E}$ is orthonorm

$\Rightarrow \langle Uv, Uv\rangle_2 = \langle v, v\rangle_1$

$\Downarrow$

U is an isometry. ①

$\forall w \in V_2,\ \exists!\, \beta_1, \dots, \beta_n \in \mathbb{C},\ w = \sum_{k=1}^{n} \beta_k f_k$

let $u := \sum_{k=1}^{n} \beta_k e_k \in V_1$, so $Uu = U\sum_{k=1}^{n} \beta_k e_k$

$$= \sum_{k=1}^{n} \beta_k \underbrace{Ue_k}_{f_k}$$

$$= w$$

$\Rightarrow U$ is onto ②

① & ② $\Rightarrow U$ is a unitary operator.

Problem 3 Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional complex inner-product space and $L, J : V \to V$ be linear operators with domain V.

3.a (10 points) Show that $(LJ)^* = J^*L^*$.

$\forall u, v \in V, \quad \langle u, LJv\rangle = \langle (LJ)^* u, v\rangle$

Also $\quad \langle u, LJv\rangle = \langle L^* u, Jv\rangle = \langle J^*L^*u, v\rangle$

$\Rightarrow \quad \langle (LJ)^* u, v\rangle = \langle J^*L^* u, v\rangle$

$\Rightarrow \quad \langle ((LJ)^* - J^*L^*)u, v\rangle = 0$

choose $v = ((LJ)^* - J^*L^*)u \Rightarrow \langle v, v\rangle = 0 \Rightarrow v = 0$

$\Rightarrow ((LJ)^* - J^*L^*)u = 0 \Rightarrow (LJ)^* u = (J^*L^*)u \quad \forall u \in V$

$\Downarrow$

$(LJ)^* = J^*L^*$. ☒

3.b (10 points) Show that $\mathrm{tr}(LJ) = \mathrm{tr}(JL)$. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be an orthonormal basis

$\mathrm{tr}(LJ) = \sum_{i=1}^{n} \langle e_i, LJe_i\rangle \qquad Je_i = \sum_{j=1}^{n} \langle e_j, Je_i\rangle e_j$

$\Rightarrow \quad = \sum_{i=1}^{n} \langle e_i, \underbrace{L\left(\sum_{j=1}^{n} \langle e_j, Je_i\rangle e_j\right)}_{\sum_{j=1}^{n} \langle e_j, Je_i\rangle Le_j} \rangle$

$\overset{①}{=} \sum_{i,j=1}^{n} \langle e_j, Je_i\rangle \langle e_i, Le_j\rangle \qquad$ Relabel $i \leftrightarrow j$

$= \sum_{i,j=1}^{n} \langle e_i, Je_j\rangle \langle e_j, Le_i\rangle$

$= \sum_{i,j=1}^{n} \langle e_j, Le_i\rangle \langle e_i, Je_j\rangle$

by ① $\quad = \mathrm{tr}(JL)$. ☒.

Problem 4 Let $(V, \langle\cdot,\cdot\rangle)$ be a complex inner-product space and $P : V \to V$ is a projection operator (that is not necessarily orthogonal.)

4.a (5 points) Give the definition of a normal operator $N : V \to V$ with domain V.

N is normal if $[N, N^*] = 0$.

4.b (5 points) Give the statement of the spectral theorem for normal operators acting in finite-dimensional inner-product spaces. Given a normal operator $N : V \to V$ defined on a finite-dim complex inner-product space V, there are complex numbers $\nu_1, \dots, \nu_n$ and a complete orthonormal system $\{P_1, \dots P_n\}$ of orthogonal projection operators such that

$$N = \sum_{i=1}^{n} \nu_i P_i .$$

4.c (10 points) Show if P is a normal operator, then it is Hermitian.

By spectral theorem $\exists\, \pi_1, \dots, \pi_n \in \mathbb{C}$ & a $\{P_1, \dots, P_n\}$ as above such that

$$P = \sum_{i=1}^{n} \pi_i P_i$$

$$\Rightarrow \quad P^2 = \Big(\sum_{i=1}^{n} \pi_i P_i\Big)\Big(\sum_{j=1}^{n} \pi_j P_j\Big)$$

$$= \sum_{i,j=1}^{n} \pi_i \pi_j \underbrace{P_i P_j}_{\delta_{ij} P_j} = \sum_{i=1}^{n} \pi_i^2 P_i$$

$$P^2 = P \;\Rightarrow\; \sum_{i=1}^{n} (\pi_i^2 - \pi_i) P_i = 0$$

Apply P_u for $u \in \{1, \dots, n\}$ $\Rightarrow P_u \sum_{i=1}^{n} \pi_i(\pi_i - 1) P_i = 0$

$$\Rightarrow \sum_{i=1}^{n} \pi_i(\pi_i - 1) \underbrace{P_u P_i}_{\delta_{ui} P_i} = 0 \;\Rightarrow\; \pi_u(\pi_u - 1) P_u = 0$$

$P_u \neq 0 \Rightarrow$ either $\pi_u = 0$ or $\pi_u = 1$ $\Rightarrow \pi_u \in \mathbb{R} \Rightarrow$

$P = \sum_{i=1}^{n} \pi_i P_i$ is Hermitian.

Problem 5 Consider a quantum system whose state vectors belong to a finite-dimensional inner-product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. A state of this system is a linear operator $\omega : \mathfrak{A} \to \mathbb{R}$ which maps the algebra of observables $\mathfrak{A}$ to $\mathbb{R}$ and satisfies $\omega(I) = 1$ and $\omega(A^2) \geq 0$, where I is the identity operator acting in $\mathscr{H}$ and A is an arbitrary observable.
Note: Faddeev-Yakobowskii uses the notation $\prec A|\omega \succ$ for $\omega(A)$.

5.a (15 points) Show that for any state ω, there is a unique positive operator M with unit trace such that $\omega(A) = \mathrm{tr}(MA)$.

- $\omega \in \mathfrak{A}^*$ by Riesz' Lemma $\exists! M \in \mathfrak{A} \ni \forall A \in \mathfrak{A}$

$\omega(A) = \langle M, A\rangle = \mathrm{tr}(MA)$

- $\mathrm{tr}(M) = \mathrm{tr}(MI) = \omega(I) = 1$

- $M^\dagger = M \Rightarrow \exists$ an orthonormal basis $\{e_1, \ldots, e_n\}$ consisting of the eigenvectors of M and

$\forall i \in \{1, \ldots, n\}$, $P_i := |e_i\rangle\langle e_i| \Rightarrow P_i^2 = P_i$ & $P_i^\dagger = P_i$
$\Rightarrow P_i \in \mathfrak{A}$

$$\Rightarrow \quad 0 \leq \omega(P_i^2) = \omega(P_i)$$
$$= \mathrm{tr}(MP_i)$$
$$= \sum_{j=1}^{n} \langle e_j, M \underbrace{P_i e_j}_{\delta_{ij} e_j} \rangle$$
$$= \langle e_i, M e_i \rangle$$
$$\equiv \langle e_i, \mu_i e_i \rangle$$

where μ_i is the eigenvalue of M with eigenvector e_i

$$= \mu_i \underbrace{\langle e_i, e_i \rangle}_{1}$$
$$= \mu_i$$

$\Rightarrow$ Eigenvalues of M are non-negative $\Rightarrow M$ is a positive operator. ☒

5.b (15 points) Let M_1, M_2 be the positive unit-trace operators corresponding to a pair of pure states such that $\mathrm{tr}(M_1 M_2) = 0$, ω be a convex combination of these states, i.e., it is given by the linear operator $M := (1-\alpha)M_1 + \alpha M_2$ for some $\alpha \in [0,1]$, and $A = M_1 - M_2$. Find the expectation value and the variance (standard deviation or uncertainty) of A in the state ω.

$$\langle A|\omega\rangle = \mathrm{tr}(MA)$$

$$= \mathrm{tr}\left(\left[(1-\alpha)M_1 + \alpha M_2\right](M_1 - M_2)\right)$$

$$= \mathrm{tr}\left((1-\alpha)\underbrace{M_1^2}_{=M_1} - (1-\alpha)M_1M_2 + \alpha M_2M_1 - \alpha \underbrace{M_2^2}_{=M_2}\right)$$

$$= (1-\alpha)\underbrace{\mathrm{tr}(M_1)}_{\to 1} - (1-\alpha)\underbrace{\mathrm{tr}(M_1M_2)}_{\to 0} + \alpha \underbrace{\mathrm{tr}(M_2M_1)}_{=\mathrm{tr}(M_1M_2) \to 0} - \alpha \underbrace{\mathrm{tr}\, M_2}_{\to 1}$$

$$= 1 - 2\alpha$$

$$A^2 = (M_1 - M_2)(M_1 - M_2) = \underbrace{M_1^2}_{=M_1} - M_1M_2 - M_2M_1 + \underbrace{M_2^2}_{=M_2}$$

$$= M_1 - M_1M_2 - M_2M_1 + M_2$$

$$\langle A^2|\omega\rangle = \mathrm{tr}(MA^2) = \mathrm{tr}\left(\underbrace{\left((1-\alpha)M_1 + \alpha M_2\right)\left(M_1 + M_2 - M_1M_2 - M_2M_1\right)}\right)$$

$$= \mathrm{tr}\Big[(1-\alpha)M_1 + (1-\alpha)M_1M_2 - (1-\alpha)M_1M_2 - (1-\alpha)M_1M_2M_1 + \alpha M_2M_1 + \alpha M_2 - \alpha M_2M_1M_2 - \alpha M_2M_1\Big]$$

$$= (1-\alpha) - (1-\alpha)\underbrace{\mathrm{tr}(M_1M_2M_1)}_{\mathrm{tr}(M_2M_1^2) \;=\; \mathrm{tr}(M_2M_1) = \mathrm{tr}(M_1M_2) = 0} + \alpha - \alpha \underbrace{\mathrm{tr}(M_2M_1M_2)}_{0}$$

$$= 1 - \alpha + \alpha = 1$$

$$\Rightarrow \Delta_\omega A = \sqrt{1 - (1-2\alpha)^2} = \sqrt{2\alpha(2-2\alpha)} = 2\sqrt{\alpha(1-\alpha)}$$