1) Show that every 3×3 Hermitian matrix is a linear combination of the 3×3 identity matrix $\mathbf{I}$ and the Gell-Mann matrices, i.e.,

$$\boldsymbol{\lambda}_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}_2 := \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\lambda}_4 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}_5 := \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}_6 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{\lambda}_7 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}_8 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

2) Show that for all $i, j, k \in \{1, 2, 3\}$, there are real numbers f_{ijk} such that the Gell-Mann matrices satisfy

$$[\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j] = 2i \sum_{k=1}^{8} f_{ijk} \boldsymbol{\lambda}_k.$$

Determine f_{ijk}. Notice that you only need to consider the cases that $i < j$.

3) Consider a quantum system whose state vectors belong to the three-dimensional complex Euclidean space, i.e., $\mathscr{H} = \mathbb{E}^3$. Let $\psi := (\alpha, \beta, \gamma) \in \mathscr{H} \setminus \{0\}$ and P_ψ be the corresponding orthogonal projection operator, i.e., $P_\psi := |\psi\rangle\langle\psi / \langle\psi|\psi\rangle$.

a. Show that P_ψ has two eigenvalues, namely 0 and 1, which respectively have the geometric multiplicities 2 and 1.

b. Compute the matrix representation of ψ and P_ψ in the standard basis of $\mathbb{E}^3$ that we denote by $\{e_1, e_2, e_3\}$. Recall that $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, where δ_{ij} stands for the Kronecker delta symbol.

c. Suppose that $\alpha = 1$. Then $\{\psi, e_2, e_3, \}$ forms a basis of $\mathbb{E}^3$. Construct an orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$ of $\mathbb{E}^3$ by performing Gram-Schmidt process on $\{\psi, e_2, e_3, \}$.

d. Show that for all $i \in \{1, 2, 3\}$, ψ_i is an eigenvector of P_ψ.

e. Express the representation $\mathbf{P}_\psi$ of P_ψ you find in part b in terms of the Gell-Mann matrices, i.e., find $x_1, x_2, \cdots, x_8 \in \mathbb{R}$ in terms of β and γ, such that

$$\mathbf{P}_\psi = \frac{1}{3}\mathbf{I} + \sum_{i=1}^{8} x_i \boldsymbol{\lambda}_i.$$

f. Study the consequences of the statement of part d of this problem for $x_1, x_2, \cdots, x_8$. In particular, try to find algebraic equations satisfied by $x_1, x_2, \cdots, x_8$ that signify the subset of $\mathbb{R}^8$ signifying the set of pure states of the system that are not orthogonal to e_1.