Consider a quantum system whose state vectors belong to the 2-dimensional complex Euclidean space $\mathbb{E}^2$. At $t = 0$, the system is in the state given by the density operator

$$M_0 := \frac{1}{4}(P_\psi + 3P_\phi),$$

where

$$\psi := \frac{1}{\sqrt{2}}(1,1), \qquad \phi := \frac{1}{\sqrt{2}}(1,-i), \qquad P_\psi := |\psi\rangle\langle\psi|, \qquad P_\phi := |\phi\rangle\langle\phi|.$$

The system's dynamics is determined by the Hamiltonian operator,

$$H := E\big(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|\big),$$

where $E \in \mathbb{R}^+$ and $\{e_1, e_2\}$ is the standard basis of $\mathbb{C}^2$, i.e., $e_1 := (1,0)$ and $e_2 := (0,1)$. At $t = T := \pi E/4\hbar$, an observer measures the observable

$$A := a\big(|\alpha_1\rangle\langle\alpha_1| - 2|\alpha_2\rangle\langle\alpha_2|\big),$$

where $a \in \mathbb{R}^+$, $\alpha_1 := (2,i)/\sqrt{5}$ and $\alpha_2 := (i,2)/\sqrt{5}$.

1) Find the matrix representation of the following quantities in the standard basis of $\mathbb{C}^2$: ψ, ϕ, P_ψ, P_ϕ, M_0, e_1, e_2, H, α_1, α_2, and A.

2) Find the eigenvalues and eigenvectors of M_0, H, and A. Explain why ϕ and ψ are not eigenvectors of M_0. Notice that the eigenvectors are elements of $\mathbb{C}^2$, i.e., they have the form (z_1, z_2) for a pair of complex numbers z_1 and z_2.

3) Find the explicit form of the time-evolution operator $U(t,t_0) := e^{-i(t-t_0)H/\hbar}$ and its matrix representation in the standard basis of $\mathbb{C}^2$.

4) Find the explicit form of the standard matrix representation $\mathbf{M}(t)$ of the density operator $M(t)$ that describes the evolving state of the system for $t \in (0,T)$ (in the Schrödinger picture of dynamics).

5) Find the explicit form of the standard matrix representation $\mathbf{A}(t)$ of evolving observable $A(t)$ with $A(0) = A$ for $t \in (0,T)$ (in the Heisenberg picture of dynamics).

6) Compute the expectation value and the uncertainty (variance) of A for $t \in (0,T)$ in the Schrödinger picture of dynamics.

7) Compute the expectation value and the uncertainty (variance) of $A(t)$ for $t \in (0,T)$ in the Heisenberg picture of dynamics.

8) What is the probability of finding each of the values: $-2a, -a, 0, a$, and $2a$, by measuring A at $t = T$.

9) Find the explicit form of the standard matrix representation $\mathbf{M}(t)$ of the density operator $M(t)$ that describes the evolving state of the system for $t \in (T, 2T)$ (in the Schrödinger picture of dynamics). Notice that this depends on the outcome of the measurement of A at $t = T$. Therefore you must produce separate formulas for the possible outcomes (each having certain probability to be realized.)

10) What is the probability of finding each of the values: $-2a, -a, 0, a$, and $2a$, by measuring A at $t = 2T$.

11) What is the expectation value and the uncertainty of A at $t = 2T$?