Math 450/586: Final Exam

Fall 2009

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours and 45 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

To be filled by the grader:

Grade:	

Problem 1. Give the definition of the following concepts. (20 points)

1.a) A tensor of type (r, s) for a finite-dimensional real vector space:

1.b) Tensor product of two covariant tensors:

1.c) A basis of a topological space:

1.d) A locally Euclidean topological space:

1.e) A C^∞-atlas and a C^∞-manifold:

1.f) Tangent vectors and tangent spaces of a C^∞-manifold:

1.g) Grassmann algebra of a C^∞-manifold:

1.h) A Riemannian manifold:

Problem 2. Let ω be a C^∞-one-form and X be a C^∞-vector field defined on a C^∞-manifold M. Use local coordinate representation of ω and X in a coordinate chart of M to prove the identity:

$$d(i_X\omega) + i_X(d\omega) = \mathcal{L}_X\omega.$$

Here d, i_X, and $\mathcal{L}_X$ stand for the exterior derivative, interior product, and the Lie derivative. (20 points)

Problem 3. Let (r, θ) be the polar coordinates on $\mathbb{R}^2$, (x, y, z) be Cartesian coordinates on $\mathbb{R}^3$, and $x^1 := r$, $x^2 := \theta$, $y^1 := x$, $y^2 := y$, $y^3 := z$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(x^1, x^2) := (x^1 \cos x^2, x^1 \sin x^2, \sqrt{3}\, x^1)$, let

$$\begin{aligned} O &:= \left\{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 > 0, -\frac{\pi}{2} < x^2 < \frac{3\pi}{2}\right\}, \\ M &:= \operatorname{Im}(O) = \left\{F(x^1, x^2) \mid x^1 > 0, -\frac{\pi}{2} < x^2 < \frac{3\pi}{2}\right\}. \end{aligned}$$

and $\phi := F^{-1} : M \to O$. Then M is a C^∞-manifold and $\{(M, \phi)\}$ is a C^∞-atlas for M.

3.a) Let $\imath : M \to \mathbb{R}^3$ be the inclusion map, i.e., $\imath(p) := p$ for all $p \in M$, and $\imath_* : TM_p \to T\mathbb{R}^3_p$ be the differential of $\imath$ at p. Compute $\imath_*(\frac{\partial}{\partial x^1})$ and $\imath_*(\frac{\partial}{\partial x^2})$ in the coordinate chart (M, ϕ). (10 points)

Hint: Note that $\imath_*(\frac{\partial}{\partial x^i}) = V_i^j \frac{\partial}{\partial y^j}$. You should determine the components V_i^j of $\imath_*(\frac{\partial}{\partial x^i})$.

3.b) Let $g_E := \sum_{i=1}^{3} dy^i \otimes dy^i$ be the Euclidean metric tensor on $\mathbb{R}^3$ and $g := \imath^*(g_E)$ be the pullback of g_E onto M. Compute the components of g in the coordinate chart (M, ϕ), i.e., write $g = g_{jk}\, dx^j \otimes dx^j$ and determine g_{jk} for all $j, k \in \{1, 2\}$. (15 points)

Hint: Check that $\Gamma_{jk\ell} := \frac{1}{2}(g_{jk,\ell} + g_{j\ell,k} - g_{k\ell,j})$ have the following values.

$$\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = 0, \quad \Gamma_{122} = -x^1, \quad \Gamma_{212} = \Gamma_{221} = x^1.$$

3.c) Let $\gamma : [0, \pi] \to M$ be the curve defined by $\gamma(t) = (\cos t, \sin t, \sqrt{3})$. Use the Levi Civita connection defined by g (i.e., the symmetric connection that is compatible with g) to parallel transport the vector $\frac{\partial}{\partial x^1}$ from $\gamma(0)$ to $\gamma(\pi)$ along γ. You must express the parallel transported vector in the form $V^i(t)\frac{\partial}{\partial x^i}$ and determine $V^i(\pi)$. Note that you may use the values of Γ_{ijk} given in problem 3.b. (15 points)

Problem 4. Let M be a two-dimensional Riemannian manifold. Suppose that in some local coordinate chart (U_α, ϕ_α) the components of the metric tensor are given by

$$g_{11}(x^1, x^2) = g_{22}(x^1, x^2) = f(x^1, x^2), \quad g_{12}(x^1, x^2) = g_{21}(x^1, x^2) = 0, \qquad \forall (x^1, x^2) \in U_\alpha,$$

where $f : U_\alpha \to \mathbb{R}^+$ is a C^∞-function.

4.a) Express the Christoffel symbols for the Levi Civita connection in terms of f and its partial derivatives. Use the notation $f_{,j}$ for $\frac{\partial f}{\partial x^j}$ and simplify your result as much as possible. (10 points)

4.b) Compute R^1_{212}. (10 points)

Hint: $R^\ell_{ijk} = \Gamma^\ell_{ik,j} - \Gamma^\ell_{ij,k} + \Gamma^\ell_{jm}\Gamma^m_{ki} - \Gamma^\ell_{km}\Gamma^m_{ji}$. (10 points)