Solution

Math 503: Quiz # 3

Fall 2006

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- You have 60 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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Problem 1. Solve the following Sturm-Liouville problem.

$$y'' + \lambda y = 0, \quad y'(-\pi) = 0, \quad y'(\pi) = 0,$$

i.e., find all real values of λ for which a solution exists and determine this solution. (40 points)

Consider the following case:

(1) $\lambda \in \mathbb{R}^- \Rightarrow \exists \mu \in \mathbb{R}^+, \ \lambda = -\mu^2 \Rightarrow$

$$y'' - \mu^2 y = 0 \Rightarrow y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$\Rightarrow y' = \mu (c_1 e^{\mu x} - c_2 e^{-\mu x})$$

$$0 = y'(-\pi) = \mu (c_1 e^{-\pi\mu} - c_2 e^{\pi\mu})$$
$$0 = y'(\pi) = \mu (c_1 e^{\pi\mu} - c_2 e^{-\pi\mu})$$
$\mu \neq 0 \Rightarrow$

$$c_1 = e^{2\pi\mu} c_2, \quad c_1 = e^{-2\pi\mu} c_2 \Rightarrow$$

$$c_2 = e^{4\pi\mu} c_2 \Rightarrow (1 - e^{4\pi\mu}) c_2 = 0 \Rightarrow \text{either } c_2 = 0 \Downarrow c_1 = 0 \Downarrow y \equiv 0 \Downarrow \text{not accepted}$$

or $e^{4\pi\mu} = 1 \Rightarrow \mu = 0.$ $\Downarrow$ not the case

So case (1) does not arise. 10

(2) $\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y = c_1 + c_2 x$

$\Rightarrow y' = c_2$

$0 = y'(-\pi) = c_2 \Rightarrow c_2 = 0 \Rightarrow y'(\pi) = 0$ ✓

So in this case $\boxed{y(x) = c_1 \quad \text{when } c_1 \in \mathbb{R} - \{0\}.}$

This is an acceptable solution it corresponds to

$\boxed{\lambda = 0}$ (10)

(3) $\lambda \in \mathbb{R}^+ \Rightarrow \exists \mu \in \mathbb{R}^+, \ \lambda = \mu^2 \Rightarrow y'' + \mu^2 y = 0$

$\Rightarrow y = c_1 e^{i\mu x} + c_2 e^{-i\mu x}$

$\Rightarrow y' = i\mu (c_1 e^{i\mu x} - c_2 e^{-i\mu x})$

$0 = y'(-\pi) = i\mu (c_1 e^{-i\pi\mu} - c_2 e^{i\pi\mu})$ $\quad \mu \neq 0 \Rightarrow$

$0 = y'(\pi) = i\mu (c_1 e^{i\pi\mu} - c_2 e^{-i\pi\mu})$

$c_1 = e^{2i\pi\mu} c_2 \quad , \quad c_1 = e^{-2i\pi\mu} c_2$

$\Rightarrow c_2 = e^{4i\pi\mu} c_2 \Rightarrow (1 - e^{4i\pi\mu}) c_2 = 0;$

$c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow y = 0 \Rightarrow$ not accepted.

$c_2 \neq 0 \Rightarrow e^{4i\pi\mu} = 1 \Rightarrow 4i\pi\mu = 2\pi i n$

for some $n \in \mathbb{Z}^+$

$\Rightarrow \mu = \frac{n}{2}$ for $\boxed{\text{some } n \in \mathbb{Z}^+.}$

(10) $\Rightarrow \boxed{\lambda = \frac{n^2}{4}}$ & $y(x) = c_2 (e^{2i\pi\mu} e^{i\mu x} + e^{-i\mu x})$

$\Rightarrow y(x) = c_2 (e^{i\pi n} e^{\frac{inx}{2}} + e^{-\frac{inx}{2}}) = c_2 ((-1)^n e^{\frac{inx}{2}} + e^{-\frac{inx}{2}})$

$= \underbrace{2(-1)^n c_2}_{k_n} \left[\frac{e^{\frac{inx}{2}} + (-1)^n e^{-\frac{inx}{2}}}{2} \right]$

$$\boxed{y(x) = \begin{cases} k_n \cos(\frac{nx}{2}) & \text{for } n: \text{even} \\ ik_n \sin(\frac{nx}{2}) & \text{for } n: \text{odd} \end{cases}}$$ (10)

Problem 2. Find all stationary points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sin(xy)$ and determine its local minimum, maximum, and saddle points. (30 points)

$\vec{\nabla} f = 0$

$\frac{\partial f}{\partial x} = y \cos(xy)$, $\frac{\partial f}{\partial y} = x \cos(xy)$

$\frac{\partial f}{\partial x} = 0 \Rightarrow y \cos(xy) = 0$ (i)

$\frac{\partial f}{\partial y} = 0 \Rightarrow x \cos(xy) = 0$ (ii)

Either (1) $\boxed{x = 0} \Rightarrow \cos(xy) = 1 \overset{(i)}{\Rightarrow} \boxed{y = 0}$ 5

(2) $\boxed{x \neq 0} \overset{(ii)}{\Rightarrow} \cos(xy) = 0 \Rightarrow xy = (2k+1)\frac{\pi}{2}$ $k \in \mathbb{Z}$

$\Rightarrow \boxed{y = \frac{(2k+1)\pi}{2x}}$ 5

So the stationary points are those with coordinates:

$(0, 0)$ and $(x, \frac{(2k+1)\pi}{2x})$ with $x \in \mathbb{R} - \{0\}$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -y^2 \sin(xy)$ $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \cos(xy) - xy \sin(xy)$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -x^2 \sin(xy)$ $\mathbb{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ 5

- At $(0, 0)$: $\sin(xy) = 0$, $\cos(xy) = 1 \Rightarrow \mathbb{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 5

$\Rightarrow \det \mathbb{H} = -1 \Rightarrow$ Eigenvalues have opposite sign $\Rightarrow$ $\boxed{(0,0) \text{ is a saddle point}}$ 5

- A $(x, \frac{(2k+1)\pi}{2x})$ with $x \in \mathbb{R} - \{0\}$

$\cos(xy) = 0$, $\sin(xy) = (-1)^k$, $xy = \frac{(2k+1)\pi}{2}$

$$\Rightarrow \mathbb{H} = \begin{pmatrix} -\frac{(2k+1)^2 \pi^2 (-1)^k}{4x^2} & -\frac{(2k+1)\pi}{2}(-1)^k \\ -\frac{(2k+1)\pi}{2}(-1)^k & -x^2(-1)^k \end{pmatrix}$$

$\Rightarrow \det \mathbb{H} = 0 \Rightarrow$ One of the eigenvalues is zero. 5

The 2nd derivative test is <u>inconclusive</u>. Looking closely into the 3rd order corrections to linear approximation of f shows that $(x, \frac{(2k+1)\pi}{2x})$ is neither min, max, nor a saddle point.

3rd Order Corrections: (Not Expected in Quiz)

Let $a_k := \frac{(2k+1)\pi}{2}$

$$\mathbb{H} = (-1)^{k+1}\begin{pmatrix} \frac{a_k^2}{x^2} & a_k \\ a_k & x^2 \end{pmatrix}$$

$$\det(\mathbb{H} - h\,\mathbb{I}) = \det\begin{bmatrix} (-1)^{k+1}\frac{a_k^2}{x^2} - h & (-1)^{k+1} a_k \\ (-1)^{k+1} a_k & (-1)^{k+1} x^2 - h \end{bmatrix}$$

$$= a_k^2 - (-1)^{k+1} x^2 h - (-1)^{k+1}\frac{a_k^2}{x^2} h + h^2 - a_k^2 = 0$$

$$\Rightarrow \quad h\left[h - (-1)^{k+1}\left(x^2 + \frac{a_k^2}{x^2}\right)\right] = 0$$

$$\Rightarrow \quad h_1 = 0 \quad \& \quad h_2 = (-1)^{k+1}\left(x^2 + \frac{a_k^2}{x^2}\right)$$

$h = h_1 = 0$

$$\mathbb{H}\vec{v}_1 = \vec{0} \Rightarrow (-1)^{k+1}\begin{pmatrix} \frac{a_k^2}{x^2} & a_k \\ a_k & x^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \alpha = x^2\gamma_1, \quad \beta = -a_k\gamma_1 \quad \Rightarrow \quad \vec{v}_1 = \gamma_1\begin{pmatrix} x^2 \\ -a_k \end{pmatrix}$$

Along $\vec{v}_1$ direction: $\quad \vec{\Delta x} = \varepsilon\begin{pmatrix} x^2 \\ -a_k \end{pmatrix} \qquad \varepsilon \in \mathbb{R}$

$$\sin[(x+\varepsilon\Delta x)(y+\varepsilon\Delta y)] = \sin(xy) + \overset{0}{\vec{\nabla} f\cdot\vec{\Delta x}} + \frac{1}{2}\underbrace{\vec{\Delta x}\cdot\mathbb{H}\vec{\Delta x}}_{0} +$$

$$+ \frac{1}{6}\left(f_{xxx}(\Delta x)^3 + f_{yyy}(\Delta y)^3 + 3 f_{xxy}(\Delta x)^2\Delta y + 3 f_{yyx}\Delta x(\Delta y)^2\right) + \cdots$$

$$f_{xxx} = -y^3\cos(xy) = 0, \qquad f_{yyy} = -x^3\cos(xy) = 0$$

$$f_{xxy} = -\left[2y\sin(xy) + xy^2\cos(xy)\right] = -2y(-1)^k = \frac{(2k+1)\pi(-1)^{k+1}}{x}$$

$$f_{yyx} = -\left[2x\sin(xy) + yx^2\cos(xy)\right] = -2x(-1)^k = x(-1)^{k+1}$$

$$\sin[(x+\varepsilon\Delta x)(y+\varepsilon\Delta y)] = (-1)^k + \frac{\varepsilon^3}{2}\left[\frac{(2k+1)\pi(-1)^{k+1}}{x}\cdot x^4\left(-\frac{(2k+1)\pi}{2}\right) + x(-1)^{k+1}\cdot\frac{(2k+1)^2\pi^2}{4}\cdot x^2\right] + \cdots$$

$\Rightarrow$ Along $\vec{v}_1$ direction

$$\sin\left[(x+\epsilon\Delta x)(y+\epsilon\Delta y)\right] = (-1)^k + \Bigg(\overbrace{\frac{(-1)^k(2k+1)^2\pi^2x^3}{8}}^{\neq 0}\Bigg)\epsilon^3 + \cdots$$

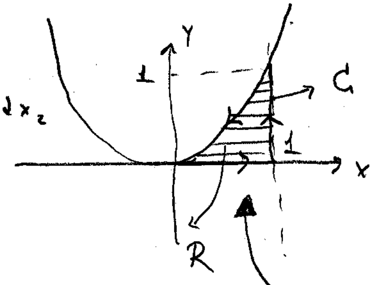
This shows that along $\vec{v}_1$, (x,y) is an inflection point along the normal direction $\vec{v}_2$ it behaves as a min for k: odd & max for k: even.

So these points are neither min nor max. strictly speaking they are not saddle points either.

Problem 3. Let $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$ and $R\subset\mathbb{R}^2$ be defined by

$$\vec{F}(x,y):=\begin{pmatrix} x\cosh y \\ x^2\sinh y\end{pmatrix},\qquad R:=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le x^2\ \}.$$

Sketch the boundary curve C of R and use Green's theorem in plane to evaluate $\int_C \vec{F}(\vec{x})\cdot d\vec{x}$ counterclockwise. (30 points)



5

$$I=\int_C \vec{F}(\vec{x})\cdot d\vec{x} \overset{5}{=} \iint_R (\partial_1 F_2-\partial_2 F_1)\,dx_1\,dx_2$$

$$\partial_1 F_2=\frac{\partial}{\partial x}(x^2\sinh y) = 2x\sinh y$$

$$\partial_2 F_1=\frac{\partial}{\partial y}(x\cosh y) = x\sinh y$$

$$\Rightarrow\quad I=\int_0^1 dx\int_0^{x^2} dy\,[2x\sinh y - x\sinh y] \quad 5$$

$$=\int_0^1 dx\int_0^{x^2} dy\; x\sinh y$$

$$=\int_0^1 dx\left(x\cosh y\Big|_0^{x^2}\right)$$

$$=\int_0^1 dx\,[x(\cosh(x^2)-1)] \quad 5$$

$$=\int_0^1 x\cosh(x^2)\,dx-\int_0^1 x\,dx$$

$$u=x^2 \Rightarrow \quad du=\frac{x}{2}\,dx$$

$$=2\int_0^1 \cosh(u)\,du-\frac{x^2}{2}\Big|_0^1 \quad 5$$

$$=2\sinh(u)\Big|_0^1-\frac{1}{2} = 2\sinh(1)-\frac{1}{2}$$

$$=e-e^{-1}-\frac{1}{2} \quad 5$$