Math 503: Midterm Exam 2

Fall 2007

You have two hours.

Problem 1. Let (ρ, θ) denote the polar coordinates in $\mathbb{R}^2$, i.e., $\rho = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$, where (x, y) are the Cartesian coordinates in $\mathbb{R}^2$. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field that only depends on θ, i.e., $\mathbf{F}(\rho, \theta) = F_1(\theta)\mathbf{i} + F_2(\theta)\mathbf{j}$, where $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : \mathbb{R} \to \mathbb{R}$ are differentiable periodic functions with period 2π, and $\mathbf{i}$ and $\mathbf{j}$ are respectively the unit vectors along the x- and y-axes.

1.a) Let R be the unit disc defined by $\rho \leq 1$ and C be the unit circle defined by $\rho = 1$. Verify the statement of the Green's theorem in plane for $\mathbf{F}$ and R, i.e., evaluate the left- and right-hand sides of the formula given by Green's theorem and show that they are identical. (20 points)

1.b) Show that if $\mathbf{F}$ is conservative, one can express F_1 in terms of F_2. Obtain the expression of F_1 in terms of F_2 and use it to compute $F_1(\theta)$ if $F_2(\theta) = \cos^3 \theta$. (15 points)

Problem 2. Let (ρ, θ, z) denote the cylindrical coordinates, i.e., $\rho = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. Let S denote the paraboloid defined by $z = \frac{\rho^2}{2}$. Find the general form of the geodesics (curves of minimum distance) on S. (20 points)
Hint: Parameterize the curves on S by ρ, i.e., denote their points as $(\rho, \theta(\rho), \frac{\rho^2}{2})$ in the cylindrical coordinates and obtain $\theta(\rho)$.

Problem 3. Let K and L be differential operators defined by: $Ku := u_x + f(x, y)u_y$ and $Lu := u_x + g(x, y)u_y$, where $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions.

3.a) Find f and g such that the equation $u_{xx} - x^2 u_{yy} - u_y = 0$ can be expressed in the form $KLu = 0$. (10 points)

3.b) Use (3.a) to reduce $u_{xx} - x^2 u_{yy} - u_y = 0$ into first order equations and solve these equations to obtain the general solution of $u_{xx} - x^2 u_{yy} - u_y = 0$. (20 points)

Problem 4. Solve the following initial-boundary-value problem. (15 points)

$$
\begin{aligned}
&u_t(x, t) = u_{xx}(x, t) + u(x, t), \qquad 0 \leq x \leq \pi, \quad t \geq 0 \\
&u(0, t) = u(\pi, t) = 0, \qquad t \geq 0. \\
&u_t(x, 0) = \sin(2x) - \sin(3x), \qquad 0 \leq x \leq \pi.
\end{aligned}
$$