Math 503: Quiz # 4

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 60 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{F}(x, y, z) := (\sin(x - z) - 3y^3, \cos(y - z) + 3x^3, \ln(1 + x^2 + y^2))$ for all $(x, y, z) \in \mathbb{R}^3$, and $C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \quad z = 1\}$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ counterclockwise along C. (25 points)

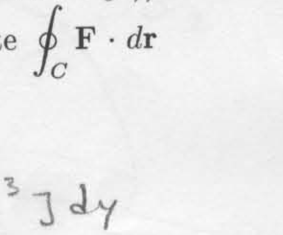
Along C, $z = 1 \Rightarrow dz = 0 \Rightarrow$

$$I := \oint_C \vec{F} \cdot d\vec{r} = \oint_C [\sin(x-1) - 3y^3]\,dx + [\cos(y-1) + 3x^3]\,dy$$

$$= \oint_{\tilde{C}} [\sin(x-1) - 3y^3]\,dx + [\cos(y-1) + 3x^3]\,dy$$

where $\tilde{C}$ is the unit circle $x^2 + y^2 = 1$ in the x-y plane. By Green's thm:

$$I = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \qquad \text{where } S \text{ is the disc } x^2+y^2 \leq 1$$



$$= \iint_S (9x^2 + 9y^2)\,dx\,dy = 9 \int_0^{2\pi} d\theta \int_0^1 r^3\,dr$$

$$= 9(2\pi)\left(\frac{r^4}{4}\Big|_0^1\right) = \frac{9\pi}{2}.$$

2. Consider the surfaces

$$S_1 := \{(x,y,z) \in \mathbb{R}^3 \mid z = 1 - \sqrt{x^2+y^2},\ \ 0 \le z \le 1\},$$

$$S_2 := \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{1-(x^2+y^2)},\ \ 0 \le z \le 1\},$$

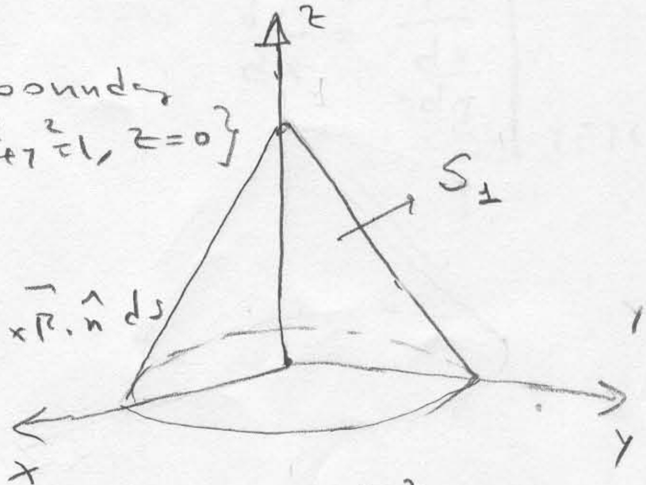
and let $\mathbf{r} := (x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$, $r := |\mathbf{r}|$, $\hat{\mathbf{r}} := \frac{\mathbf{r}}{r}$ for $r \neq 0$, and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function such that $(\nabla \times \mathbf{F})(x,y,z) = f(r)\hat{\mathbf{r}}$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$.

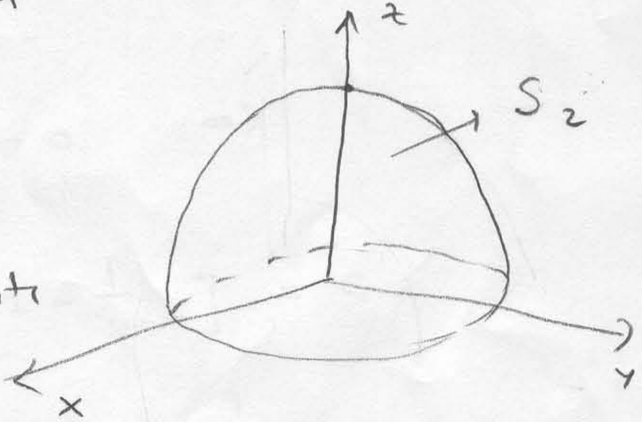
2.a) Show that $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is the unit outward normal vector to the corresponding surface. (5 points)

S_1 & S_2 have the same boundary namely $C := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1,\ z=0\}$

Therefore by Stokes' thm.

$$\iint_{S_1} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds$$





2.b) Evaluate $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. (20 points)

Using (2.a) we can instead compute $\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds$. The result is:

$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \iint_{S_2} f(r)\, \hat{n} \cdot \hat{r}\, ds$$

$$= \iint_{S_2} f(1) \cdot dS = f(1) \underbrace{\iint_{S_2} ds}_{\text{Area of } S_2 = 2\pi}$$

$$= 2\pi f(1).$$

3. Find the stationary points of the following functional with fixed boundary conditions. (30 points)

$$\mathcal{F}[y(x)] := \int_0^1 \frac{1+y'^2}{1+y^2}\,dx.$$

Hint: $\int \frac{dt}{\sqrt{1+t^2}} = \sinh^{-1}(t) + c.$

$$F = \frac{1+y'^2}{1+y^2} \quad \text{view} \quad x = x(y) \Rightarrow \frac{dx}{dy} = \frac{1}{dy/dx}\bigg|_{x=x(y)} \Leftrightarrow \frac{dy}{dx} = \frac{1}{dx/dy}\bigg|_{y=y(x)}$$

$$dx = \frac{dx}{dy}\,dy = x'dy$$

$$\Rightarrow \mathcal{F} = \int_0^{-1} \left(\frac{1+\frac{1}{(x')^2}}{1+y^2} \right) x'dy = \int_0^{-1} \underbrace{\frac{x' + \frac{1}{x'}}{1+y^2}}_{F} dy \text{, and we want } \delta\mathcal{F} = 0$$

$$\Rightarrow \cancelto{0}{\frac{\partial F}{\partial x}} - \frac{d}{dy}\frac{\partial F}{\partial x'} = 0 \Rightarrow \frac{\partial F}{\partial x'} = \text{constant} = c \Rightarrow \frac{1}{1+y^2}\left[1 - \frac{1}{x'^2}\right] = c$$

$$\Rightarrow 1 - \frac{1}{x'^2} = c(1+y^2) \Rightarrow x'^2 = \frac{1}{1+c(1+y^2)} \Rightarrow x' = \frac{\pm 1}{\sqrt{(1+c)+cy^2}}$$

$$\Rightarrow x' = \frac{\pm 1}{\sqrt{1+c}\sqrt{1+\left(\frac{c}{1+c}\right)y^2}} \Rightarrow X = \frac{\pm 1}{\sqrt{1+c}} \int \frac{dy}{\sqrt{1+\left(\frac{c}{1+c}\right)y^2}}, \quad \text{let } \sqrt{\frac{c}{1+c}}\,y = \sinh\theta$$

$$\Rightarrow \sqrt{1+\left(\frac{c}{1+c}\right)y^2} = \sqrt{1+\sinh^2\theta} = \cosh\theta, \quad dy = \sqrt{\frac{1+c}{c}}\cosh\theta\,d\theta$$

$$X = \frac{\pm 1}{\sqrt{1+c}}\sqrt{\frac{1+c}{c}}\int d\theta = \frac{\pm\theta}{\sqrt{c}} + K \Rightarrow \pm\sqrt{c}\,(x-K) = \theta$$

$$\Rightarrow \pm\sinh\theta[\sqrt{c}\,(x-K) = \sinh\theta = \sqrt{\frac{c}{1+c}}\,y$$

$$\Rightarrow \boxed{y = \pm\sqrt{\frac{1+c}{c}}\,\sinh[\sqrt{c}\,(x-K)]}$$

4. Evaluate the functional derivative $\frac{\delta \mathcal{F}[y(x)]}{\delta y(x')}$ of $\mathcal{F}[y(x)] := \int_0^1 \sin(y) y''\, dx$ for the cases that $y(0) = 0$ and $y(1) = \pi$. (20 points)

$$\mathcal{F}[y(x)] = \int_0^1 \left(\frac{d}{dx}[\sin y\, y'] - \cos y\, y'^2 \right) dx$$

$$= \underbrace{\sin y\, y' \Big|_0^1}_{0} - \int_0^1 (\cos y\, y'^2)\, dx$$

$$= \int_0^1 \underbrace{[-\cos y\, y'^2]}_{F}\, dx$$

$$\frac{\delta \mathcal{F}[y(x)]}{\delta y(x')} = \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \Bigg|_{x=x'} =$$

$$= \left[\sin y\, y'^2 - \frac{d}{dx} (-2\cos y\, y') \right] \Big|_{x=x'}$$

$$= \left[\sin y\, y'^2 + 2(-\sin y\, y'^2 + \cos y\, y'') \right] \Big|_{x=x'}$$

$$= \left(-\sin y\, y'^2 + 2\cos y\, y'' \right) \Big|_{x=x'}$$