Soln's

Phys 401/OEPE 541: Midterm Exam 3

December 09, 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 Consider measuring an observable $\hat{O}$ in a state λ_ψ of a quantum system. Show that

1.a (5 points) The expectation value of $\hat{O}$ in the state λ_ψ is a real number.

$\langle O\rangle_{\lambda_\psi} = \dfrac{\langle\psi|\hat{O}\psi\rangle}{\langle\psi|\psi\rangle}$ $\quad \langle\psi|\hat{O}\psi\rangle^* = \langle\hat{O}\psi|\psi\rangle = \langle\psi|\hat{O}\psi\rangle$ because O is Hermi

$\Rightarrow \langle\psi|\hat{O}\psi\rangle$ is real.

$\langle\psi|\psi\rangle$ is also real $\quad \Rightarrow \langle\hat{O}\rangle_{\lambda_\psi}$ is real.

1.b (5 points) If the uncertainty in measuring $\hat{O}$ in the state λ_ψ is zero, then ψ is an eigenvector of $\hat{O}$.

$\Delta\hat{O}_{\lambda_\psi} = \sqrt{\langle(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})^2\rangle_{\lambda_\psi}} = 0 \Leftrightarrow \langle(\hat{O} - \langle\hat{O}\rangle\hat{I}_{\lambda_\psi})^2\rangle_{\lambda_\psi} = 0 \Leftrightarrow$

$\langle\psi|(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})^2\psi\rangle = 0$ Because $\langle\hat{O}\rangle_{\lambda_\psi}$ is real & $\hat{O}$ is Hermitian, so is $\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I} \Rightarrow$

$\langle(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})\psi|(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi})\psi\rangle = 0 \Leftrightarrow (\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})\psi = 0$

$\Rightarrow \hat{O}\psi = \langle\hat{O}\rangle_{\lambda_\psi}\psi \Rightarrow \psi$ is an eigenvector of $\hat{O}$.

Problem 2 A quantum system is determined by a two-dimensional Hilbert space $\mathscr{H}$ and a Hamiltonian operator $\hat{H} := iE(|1\rangle\langle 2| - |2\rangle\langle 1|)$, where E is a positive real parameter and $\{|1\rangle, |2\rangle\}$ is an orthonormal basis of $\mathscr{H}$. Suppose that at time $t = 0$, the system is in the state determined by $|1\rangle$ and that we measure the observable $\hat{A} := |1\rangle\langle 1| - |2\rangle\langle 2|$ at times $t_1 = \pi\hbar/4E$ and $t_2 = \pi\hbar/2E$.

2.a (6 points) Find the matrix representation of $\hat{H}$ and $\hat{A}$ in the basis $\{|1\rangle, |2\rangle\}$.

$$\mathbb{H} = [\langle i|\hat{H}|j\rangle] = \begin{bmatrix} 0 & iE \\ -iE & 0 \end{bmatrix}$$

$$\mathbb{A} = [\langle i|\hat{A}|j\rangle] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2.b (10 points) Find the matrix representation of the time-evolution operator $\hat{U}(t, t_0)$ for the system in the basis $\{|1\rangle, |2\rangle\}$. Recall that $\hat{U}(t, t_0) = e^{-\frac{i(t-t_0)}{\hbar}\hat{H}}$.

To get full credit you must simplify your response as much as possible.

$$H|\psi_n\rangle = E_n|\psi_n\rangle \iff \mathbb{H}\underbrace{\begin{bmatrix} a_n \\ b_n \end{bmatrix}}_{\psi_n} = E_n\begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$\iff \begin{bmatrix} 0 & iE \\ -iE & 0 \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} = E_n\begin{bmatrix} a_n \\ b_n \end{bmatrix} \iff \begin{bmatrix} -E_n & iE \\ -iE & -E_n \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E_n^2 - E^2 = 0 \Rightarrow E_n = \pm E \Rightarrow E_1 := E, \quad E_2 := -E$$

For $n=1$: $E_1 = E \Rightarrow \begin{bmatrix} -E & iE \\ -iE & -E \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -ia_1 - b_1 = 0$

$\Rightarrow b_1 = -ia_1$ take $a_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow \psi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$ so that $\langle\psi_1|\psi_1\rangle = \psi_1^\dagger\psi_1 = 1$.

For $n=2$: $E_2 = -E \Rightarrow \begin{bmatrix} E & iE \\ -iE & E \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -ia_2 + b_2 = 0$

$\Rightarrow b_2 = ia_2$ take $a_2 = \frac{1}{\sqrt{2}}$

$\Rightarrow \psi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ again $\langle\psi_2|\psi_2\rangle = \psi_2^\dagger\psi_2 = 1$.

let $\mathbb{V} := [\psi_1, \psi_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \Rightarrow \mathbb{V}^{-1} = \mathbb{V}^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

$\mathbb{V}^\dagger\mathbb{H}\mathbb{V} = \underbrace{\begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}}_{\mathbb{H}_d} \Rightarrow \mathbb{H} = \mathbb{V}\mathbb{H}_d\mathbb{V}^\dagger \Rightarrow \mathbb{U}(t,t_0) = \mathbb{V}e^{-\frac{i(t-t_0)\mathbb{H}_d}{\hbar}}\mathbb{V}^\dagger$

$$\Rightarrow \mathbb{U}(t,t_0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{-\frac{i(t-t_0)E}{\hbar}} & 0 \\ 0 & e^{\frac{i(t-t_0)E}{\hbar}} \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

let $\varphi := \frac{(t-t_0)E}{\hbar}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{-i\varphi} & ie^{-i\varphi} \\ e^{i\varphi} & -ie^{i\varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

2.c (10 points) Compute the probability for each possible outcome of the first measurement of $\hat{A}$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\psi_0\rangle = |1\rangle \qquad \Psi_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi(t)\rangle = U(t,t_0)|\psi_0\rangle \qquad t \in [0, t_1] \qquad t_1 = \frac{\pi\hbar}{4E}$$

$$\Psi(t_1) = U(t_1, t_0)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{For } \begin{cases} t = t_1, \\ t_0 = 0 \end{cases} \quad \varphi = \frac{(t_1 - t_0)E}{\hbar} = \frac{t_1 E}{\hbar} = \frac{\pi}{4}$$

$$= \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad |\psi(t_1)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

Probability of measuring 1: $\mathcal{P}_1 = \dfrac{\| \Pi_1 |\psi(t_1)\rangle \|^2}{\| |\psi(t_1)\rangle \|^2} = \dfrac{\langle \psi(t_1)| \Pi_1 |\psi(t_1)\rangle}{\langle \psi(t_1)|\psi(t_1)\rangle}$

$$= \frac{\Psi(t_1)^\dagger \Pi_1 \Psi(t_1)}{\Psi(t_1)^\dagger \Psi(t_1)} \quad \Rightarrow \quad \mathcal{P}_1 = \frac{[1 \ \ -1]\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{[1 \ \ -1]\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2}$$

$$\Pi\,\Psi(t_1) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Probability of measuring -1: $\mathcal{P}_2 = 1 - \mathcal{P}_1 = \frac{1}{2}$.

2.d (14 points) Compute the probability for obtaining the same value in both measurements of $\hat{A}$.

If in the first measurement we measure +1 then state vector collapses to $\underbrace{\Pi_1 |\psi(t_1)\rangle}_{|\tilde{\psi}_1(t_1)\rangle} = |1\rangle\langle 1|\psi(t_1)\rangle$

$$\Rightarrow \tilde{\psi}_1(t_1) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}.$$

For $t \in [t_1, t_2]$, $|\psi(t)\rangle = \hat{U}(t, t_1)|\tilde{\psi}_1(t_1)\rangle$

$\Rightarrow |\psi(t_2)\rangle = \hat{U}(t_2, t_1)|\tilde{\psi}_1(t_1)\rangle$ For $t = t_2$, $t_0 = t_1$ $\Rightarrow \varphi = \frac{(t_2-t_1)E}{\hbar} = \frac{\pi}{4}$

$$\psi(t_2) = \begin{bmatrix}\cos\frac{\pi}{2} & \sin(\frac{\pi}{2})\\ -\sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2})\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\psi(t_1)$$

$\Rightarrow$ Probability of meas 1 in the second measurement $= P_1 = \frac{1}{2}$

& " " " -1 " " " " $= P_{-1} = \frac{1}{2}$

If in the first measurement we measure -1 then the state vector collapses to $\underbrace{\Pi_2|\psi(t_1)\rangle}_{|\tilde{\psi}_2(t_1)\rangle} = |2\rangle\langle 2|\psi(t_1)\rangle$

$$\Rightarrow \tilde{\psi}_2(t_1) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\end{bmatrix}$$

$\Rightarrow |\psi(t_2)\rangle = \hat{U}(t_2, t_1)|\tilde{\psi}_2(t_1)\rangle$

$$\psi(t_2) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$\Rightarrow$ Probability of measuring 1 in the second measurement:

$$\tilde{P}_1 = \frac{\psi(t_2)^\dagger \Pi_1 \psi(t_2)}{\psi(t_2)^\dagger\psi(t_2)} = \frac{[1\ 1]\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{[1\ 1]\begin{bmatrix}1\\1\end{bmatrix}} = \frac{[1\ 1]\begin{bmatrix}0\\1\end{bmatrix}}{2} = \frac{1}{2}$$

$\Rightarrow$ Probability of measuring -1 in the second measurement $=$

$$\tilde{P}_2 = 1 - \tilde{P}_1 = \frac{1}{2}$$

So the probability of measuring the same value in both measurements is $P_1 \times P_1 + P_2 \times \hat{P}_2 = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

Problem 3 Let $\hat{\mathscr{P}} : L^2(\mathbb{R}) \xrightarrow{} L^2(\mathbb{R})$ be the operator defined by $(\hat{\mathscr{P}}\psi)(x) = \psi(-x)$ for all $\psi \in L^2(\mathbb{R})$.

3.a (5 points) Show that $\hat{\mathscr{P}}$ is Hermitian. $\forall \psi, \varphi \in L^2(\mathbb{R})$

$$\langle \psi | \mathscr{P} \varphi \rangle = \int_{-\infty}^{\infty} \psi(x)^* (\mathscr{P}\varphi)(x)\, dx = \int_{-\infty}^{\infty} \psi(x)^* \varphi(-x)\, dx$$

let $y := -x \Rightarrow dy = -dx$

$$\Rightarrow \langle \psi | \mathscr{P} \varphi \rangle = \int_{\infty}^{-\infty} \psi(-y)^* \varphi(y)\, d(-y) = \int_{-\infty}^{\infty} \psi(-y)^* \varphi(y)\, dy$$

$$= \int_{-\infty}^{\infty} (\mathscr{P}\psi)(y)^* \varphi(y)\, dy = \langle \mathscr{P}\psi | \varphi \rangle$$

$\Rightarrow \mathscr{P}^\dagger = \mathscr{P}$.

3.b (5 points) Show that $\hat{\mathscr{P}}|x\rangle = |-x\rangle$.

$\forall \psi \in L^2(\mathbb{R})$, $(\hat{\mathscr{P}}\psi)(x) = \langle x | \hat{\mathscr{P}} \psi \rangle \hookrightarrow \boxed{\langle \psi | -x \rangle = \langle \hat{\mathscr{P}}\psi | x \rangle}$ (1)

$$\psi(-x) = \langle -x | \psi \rangle$$

let $\{\psi_n\}$ be an orthonormal basis of $L^2(\mathbb{R})$ so that

$\sum_{n=0}^{\infty} |\psi_n\rangle\langle\psi_n| = I \Rightarrow |-x\rangle = \sum_{n=0}^{\infty} |\psi_n\rangle\langle\psi_n|-x\rangle = \sum_{n=0}^{\infty} |\psi_n\rangle\langle\hat{\mathscr{P}}\psi_n|x\rangle$

$$= \sum_{n=0}^{\infty} |\psi_n\rangle\langle\psi_n|\hat{\mathscr{P}} x\rangle$$

$$= \hat{\mathscr{P}}|x\rangle$$

3.c (5 points) Show that $\hat{\mathscr{P}}|p\rangle = |-p\rangle$.

$$\langle x | -p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ix(-p)}{\hbar}} = \frac{e^{\frac{-ixp}{\hbar}}}{\sqrt{2\pi\hbar}} = \langle -x | p \rangle$$

$$\langle x | \mathscr{P} | p \rangle = \int_{-\infty}^{\infty} dy\, \langle x | \mathscr{P} | y \rangle \langle y | p \rangle = \int_{-\infty}^{\infty} dy\, \underbrace{\langle x | -y \rangle}_{\delta(x+y)} \langle y | p \rangle$$

$$= \langle -x | p \rangle$$

$$= \langle x | -p \rangle$$

$$\Rightarrow \mathscr{P}|p\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\mathscr{P}|p\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|-p\rangle = |-p\rangle$$

3.d (10 points) Show that $\hat{X}' := i\hat{\mathscr{P}}\hat{X}$ and $\hat{P}' := i\hat{\mathscr{P}}\hat{P}$ are Hermitian operators.

$$\hat{X}'^{\dagger} = (i\hat{\mathscr{P}}\hat{X})^{\dagger} = -i\hat{X}^{\dagger}\hat{\mathscr{P}}^{\dagger} = -i\hat{X}\hat{\mathscr{P}}$$

$$= -i\hat{X}\hat{\mathscr{P}}\int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = -i\int_{-\infty}^{\infty} dx\, \hat{X}|-x\rangle\langle x|$$

$$= \int_{-\infty}^{\infty} dx\, ix|-x\rangle\langle x| = \int_{-\infty}^{\infty} dx\, ix\,\hat{\mathscr{P}}|x\rangle\langle x|$$

$$= i\hat{\mathscr{P}}\int_{-\infty}^{\infty} dx\, x|x\rangle\langle x| = i\mathscr{P}\hat{X} = \hat{X}'.$$

$$\hat{P}'^{\dagger} = (i\hat{\mathscr{P}}\hat{P})^{\dagger} = -i\hat{P}\hat{\mathscr{P}} = -i\hat{P}\hat{\mathscr{P}}\int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$$

$$= -i\hat{P}\int_{-\infty}^{\infty} dp\, |-p\rangle\langle p| = -i\int_{-\infty}^{\infty} dp\, (-p)|-p\rangle\langle p|$$

$$= i\int_{-\infty}^{\infty} dp\, p\,\hat{\mathscr{P}}|p\rangle\langle p| = i\hat{\mathscr{P}}\int_{-\infty}^{\infty} dp\, p|p\rangle\langle p|$$

$$= i\hat{\mathscr{P}}\hat{P} = \hat{P}'.$$

3.e (5 points) Compute $[\hat{X}', \hat{P}']$.

$$[\hat{X}', \hat{P}'] = [i\hat{\mathscr{P}}\hat{X}, i\hat{\mathscr{P}}\hat{P}] = -\hat{\mathscr{P}}\hat{X}\hat{\mathscr{P}}\hat{P} + \hat{\mathscr{P}}\hat{P}\hat{\mathscr{P}}\hat{X}$$

$$\hat{\mathscr{P}}\hat{X} = \hat{\mathscr{P}}\int_{-\infty}^{\infty} dx\, x|x\rangle\langle x| = \int_{-\infty}^{\infty} dx\, x|-x\rangle\langle x| \qquad \text{let } x' = -x,\ dx' = -dx$$

$$= \int_{\infty}^{-\infty} (-dx')(-x')|x'\rangle\langle -x'| = -\int_{-\infty}^{\infty} dx'\, x'|x'\rangle\langle -x'|$$

$$= -\int_{-\infty}^{\infty} dx'\, x'|x'\rangle\langle x'|\hat{\mathscr{P}} = -\hat{X}\hat{\mathscr{P}}$$

Similarly $\hat{\mathscr{P}}\hat{P} = -\hat{P}\hat{\mathscr{P}}$ this follows by the same argument if we use p instead of x.

$$\Rightarrow [\hat{X}', \hat{P}'] = +\hat{\mathscr{P}}\hat{X}\hat{P}\hat{\mathscr{P}} - \hat{\mathscr{P}}\hat{P}\hat{X}\hat{\mathscr{P}} = \hat{\mathscr{P}}\underbrace{(\hat{X}\hat{P} - \hat{P}\hat{X})}_{i\hbar\hat{I}}\hat{\mathscr{P}}$$

$$= i\hbar\hat{\mathscr{P}}^2 = i\hbar\hat{I}$$

Problem 4 (20 points) Find the delta-function normalized (generalized) eigenfunctions of $\hat{P}' := i\hat{\mathscr{P}}\hat{P}$ and determine the spectrum of this operator.

Hint: Work in the position representation where the eigenvalue equation for $\hat{P}'$ involves the first derivative of the eigenfunction, and turn this into a second order differential equation.

$$\hat{P}'|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow \qquad \lambda \in \mathbb{R} \quad \text{because} \quad \hat{P}'^{\dagger} = \hat{P}'$$

$$\langle x|\hat{P}'|\lambda\rangle = \lambda \underbrace{\langle x|\lambda\rangle}_{\xi_\lambda(x)}$$

$$\langle x|i\underbrace{\hat{\mathscr{P}}\hat{P}}_{-\hat{P}\hat{\mathscr{P}}}|\lambda\rangle = -i\langle x|\hat{P}\hat{\mathscr{P}}|\lambda\rangle = -i(-i\hbar\tfrac{d}{dx})\underbrace{\underbrace{\langle x|\hat{\mathscr{P}}|\lambda\rangle}_{\langle -x|\lambda\rangle}}_{\xi_\lambda(-x)}$$

$$\Rightarrow \quad -\hbar\frac{d}{dx}\xi_\lambda(-x) = \lambda\,\xi_\lambda(x) \qquad (1)$$

$$\hbar\frac{d}{dx}\xi_\lambda(x) = \lambda\,\xi_\lambda(-x) \qquad (2)$$

$\lambda \times (1)$: $\quad -\hbar\frac{d}{dx}[\lambda\,\xi_\lambda(-x)] = \lambda^2\xi_\lambda(x)$

$(2) \Rightarrow \quad -\hbar\frac{d}{dx}[\hbar\frac{d}{dx}\xi_\lambda(x)]$

$$\Rightarrow -\hbar^2\xi_\lambda''(x) = \lambda^2\xi_\lambda(x) \Rightarrow \xi_\lambda''(x) + \frac{\lambda^2}{\hbar^2}\xi_\lambda(x) = 0$$

$$\Rightarrow \xi_\lambda(x) = \alpha_+ e^{\frac{i|\lambda|}{\hbar}x} + \alpha_- e^{-\frac{i|\lambda|}{\hbar}x}$$

substitute this in (2) $\Rightarrow$ $i|\lambda|[\alpha_+ e^{\frac{i|\lambda|x}{\hbar}} - \alpha_- e^{-\frac{i|\lambda|x}{\hbar}}] = |\lambda|[\alpha_+ e^{-\frac{i|\lambda|x}{\hbar}} + \alpha_- e^{\frac{i|\lambda|x}{\hbar}}]$

$$\Rightarrow \boxed{i\alpha_+ = \alpha_-}$$

$\alpha_+ = -i\alpha_-$ ✓

$$\Rightarrow \xi_\lambda(x) = \alpha_+[e^{\frac{i|\lambda|x}{\hbar}} + i e^{-\frac{i|\lambda|x}{\hbar}}]$$

Let $k_\lambda := \frac{|\lambda|}{\hbar} \Rightarrow \quad |\xi_\lambda\rangle = \alpha_+(|k_\lambda\rangle + i|-k_\lambda\rangle)$

$$\langle\xi_{\lambda'}|\xi_\lambda\rangle = \alpha_+\alpha_+'^{*}(\langle k_{\lambda'}| - i\langle -k_{\lambda'}|)(|k_\lambda\rangle + i|-k_\lambda\rangle)$$

$$= \alpha_+\alpha_+'^{*}[2\underbrace{\delta(k_\lambda - k_{\lambda'})}_{\hbar\,\delta(\lambda-\lambda')} + 2i\underbrace{\delta(k_\lambda + k_{\lambda'})}_{\neq 0}]$$

$\Rightarrow$ Choose $\quad \alpha_+ = \frac{1}{\sqrt{2\pi\hbar}}$

$\Downarrow$

So $\quad \xi'_\lambda(x) = \frac{1}{\sqrt{2\pi\hbar}} \left[e^{\frac{i|\lambda|x}{\hbar}} + i e^{-\frac{i|\lambda|x}{\hbar}} \right]$

$\lambda \in \mathbb{R}$

$\mathrm{Spec}(\hat{P}') = \mathbb{R}$