Solutions to

# Phys 401: Midterm Exam 3

December 08, 2018

- Write your name and Student ID number in the space provided below and sign.

| Name, Last Name: | |
|---|---|
| ID Number: | |
| Signature: | |

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

| Estimated Grade: | |
|---|---|
| Actual Grade: | |
| Adjusted Grade: | |

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**Problem 1** (10 points) Consider a quantum system determined by a separable Hilbert space $\mathscr{H}$ and a time-dependent Hamiltonian operator $\hat{H}(t)$ acting in $\mathscr{H}$. Use the time-dependent Schrödinger equation to show that if $\hat{H}(t)$ is Hermitian for all times $t$, then the time evolution operator $\hat{U}(t, t_0)$ is unitary.

$$\begin{cases} i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \\ |\psi(t_0)\rangle = |\psi_0\rangle \end{cases} \qquad |\psi(t)\rangle = \hat{U}(t,t_0)|\psi_0\rangle$$

$$\Rightarrow \begin{cases} i\hbar \frac{d}{dt}\hat{U}(t,t_0) \overset{①}{=} \hat{H}(t)\hat{U}(t,t_0) \\ \hat{U}(t_0,t_0) = \hat{I} \end{cases} \Rightarrow -i\hbar\frac{d}{dt}\hat{U}(t,t_0)^\dagger = \hat{U}(t,t_0)^\dagger \hat{H}(t)^\dagger \overset{②}{=} \hat{U}(t,t_0)^\dagger \hat{H}(t)$$

$$\Rightarrow \frac{d}{dt}\left(\hat{U}(t,t_0)^\dagger \hat{U}(t,t_0)\right) = \left[\frac{d}{dt}\hat{U}(t,t_0)^\dagger\right]\hat{U}(t,t_0) + \hat{U}(t,t_0)^\dagger \frac{d}{dt}\hat{U}(t,t_0)$$

$$= \left(\frac{1}{-i\hbar}\right)\hat{U}(t,t_0)^\dagger \hat{H}(t)\hat{U}(t,t_0) + \hat{U}(t,t_0)^\dagger\left[\frac{1}{i\hbar}\hat{H}(t)\hat{U}(t,t_0)\right]$$

$$= 0$$

$$\Rightarrow \hat{U}(t,t_0)^\dagger \hat{U}(t,t_0) = \hat{C} \;:\; \text{a constant operator} \quad \hat{U}(t,t_0)^\dagger\hat{U}(t,t_0) = \hat{I}$$

$$\Rightarrow \underbrace{\hat{U}(t_0,t_0)^\dagger}_{\hat{I}}\, \underbrace{\hat{U}(t_0,t_0)}_{\hat{I}} = \hat{C} \Rightarrow \hat{C} = \hat{I} \qquad \Downarrow$$

$$\hat{U}(t,t_0)^\dagger = \hat{U}(t,t_0)^{-1}.$$

**Problem 2** Consider a quantum system determined by a separable Hilbert space $\mathscr{H}$ and a time-dependent Hamiltonian operator $\hat{H}(t)$ acting in $\mathscr{H}$. Let $t_0$ be a real parameter marking an instant of time, $\psi_0$ be a nonzero state vector, and $\psi(t)$ be the solution of the time-dependent Schrödinger equation for this system that satisfies $\psi(t_0) = \psi_0$. Let $\hat{V}(t)$ be a time-dependent unitary operator, and $\tilde{\psi}(t) := \hat{V}(t)\psi(t)$.

**2.a** (5 points) Find a linear operator $\hat{\tilde{H}}(t) : \mathscr{H} \to \mathscr{H}$, such that

$$i\hbar\frac{d}{dt}\tilde{\psi}(t) = \hat{\tilde{H}}(t)\tilde{\psi}(t), \qquad \tilde{\psi}(t_0) = \hat{V}(t_0)\psi_0.$$

$$i\hbar\frac{d}{dt}\tilde{\psi}(t) = i\hbar\frac{d}{dt}\left[\hat{V}(t)\psi(t)\right] = i\hbar\left[\frac{d}{dt}\hat{V}(t)\right]\psi(t) + \hat{V}(t)\underbrace{\left[i\hbar\frac{d}{dt}\psi(t)\right]}_{\hat{H}(t)\psi(t)}$$

$$= \left[i\hbar\frac{d}{dt}\hat{V}(t) + \hat{V}(t)\hat{H}(t)\right]\psi(t)$$

$$= \underbrace{\left[i\hbar\frac{d}{dt}\hat{V}(t) + \hat{V}(t)\hat{H}(t)\right]\hat{V}(t)^{-1}}_{\hat{\tilde{H}}(t)}\tilde{\psi}(t)$$

$$\Rightarrow \quad \hat{\tilde{H}}(t) = \hat{V}(t)\hat{H}(t)\hat{V}(t)^{-1} + i\hbar\left[\frac{d}{dt}\hat{V}(t)\right]\hat{V}(t)^{-1}$$

$$= \hat{V}(t)\hat{H}(t)\hat{V}(t)^{\dagger} + i\hbar\left[\frac{d}{dt}\hat{V}(t)\right]\hat{V}(t)^{\dagger}$$

**2.b** (5 points) Show that $\hat{\tilde{H}}(t)$ is a Hermitian operator for all $t \in \mathbb{R}$.

$$\hat{\tilde{H}}^{\dagger}(t) = \left\{\hat{V}(t)\hat{H}(t)\hat{V}(t)^{\dagger} + i\hbar\left[\frac{d}{dt}\hat{V}(t)\right]\hat{V}(t)^{\dagger}\right\}^{\dagger}$$

$$= \hat{V}(t)^{\dagger\dagger}\hat{H}(t)^{\dagger}\hat{V}(t)^{\dagger} - i\hbar\,\hat{V}(t)^{\dagger\dagger}\left[\frac{d}{dt}\hat{V}(t)\right]^{\dagger}$$

$$= \hat{V}(t)\hat{H}(t)\hat{V}(t)^{\dagger} - i\hbar\underbrace{\hat{V}(t)\left[\frac{d}{dt}\hat{V}(t)^{\dagger}\right]}_{\underbrace{\frac{d}{dt}\Big[\underbrace{\hat{V}(t)\hat{V}(t)^{\dagger}}_{\hat{I}}\Big]}_{0}-\left(\frac{d}{dt}\hat{V}(t)\right)\hat{V}(t)^{\dagger}}$$

$$= \hat{V}(t)\hat{H}(t)\hat{V}(t)^{\dagger} + i\hbar\left[\frac{d}{dt}\hat{V}(t)\right]\hat{V}(t)^{\dagger}$$

$$= \hat{\tilde{H}}(t)$$

**2.c** (10 points) Suppose that $\hat{\tilde{H}}(t)$ happens to be time-independent, i.e., there is a Hermitian operator $\hat{\tilde{H}}_0$ such that $\hat{\tilde{H}}(t) = \hat{\tilde{H}}_0$ for all $t \in \mathbb{R}$. Express the time-evolution operator $\hat{U}(t, t_0)$ for the Hamiltonian $\hat{H}(t)$ in terms of $\hat{V}(t)$ and $\hat{\tilde{H}}_0$.

let $\hat{\tilde{U}}(t, t_0)$ be the time-evolution operator for $\hat{\tilde{H}}$. Because $\hat{\tilde{H}}$ is time-independent,

$$\hat{\tilde{U}}(t, t_0) = e^{-\frac{i(t-t_0)}{\hbar}\hat{\tilde{H}}} \quad (1)$$

$$\tilde{\psi}(t) = \hat{\tilde{U}}(t, t_0)\, \tilde{\psi}(t_0)$$

$$\Rightarrow \quad \hat{V}(t)\, \psi(t) = \hat{\tilde{U}}(t, t_0)\, \hat{V}(t_0)\, \psi(t_0)$$

$$\Rightarrow \quad \tilde{V}(t)\, \hat{U}(t, t_0)\, \psi(t_0) = \hat{\tilde{U}}(t, t_0)\, \hat{V}(t_0)\, \psi(t_0)$$

$$\Rightarrow \quad \hat{U}(t, t_0)\, \psi(t_0) = \hat{\tilde{V}}(t)^{-1}\, \hat{\tilde{U}}(t, t_0)\, \hat{V}(t_0)\, \psi(t_0)$$

Because $\psi(t_0)$ is arbitrary,

$$\Rightarrow \quad \hat{U}(t, t_0) = \hat{\tilde{V}}(t)^{-1}\, \hat{\tilde{U}}(t, t_0)\, \hat{V}(t_0) \quad (1)$$

$$= \hat{\tilde{V}}(t)^{\dagger}\, e^{-\frac{i(t-t_0)}{\hbar}\hat{\tilde{H}}}\, \hat{V}(t_0).$$

**Problem 3** A quantum system is determined by the Hilbert space $L^2(\mathbb{R})$ and the Hamiltonian $\hat{H} = \frac{1}{2}\left[\alpha\,\hat{P}^2 + \beta(\hat{X}\hat{P} + \hat{P}\hat{X}) + \gamma\,\hat{X}^2\right]$, where $\alpha, \beta$, and $\gamma$ are constant real parameters.

**3.a** (10 points) Find the explicit form of the Heisenberg equations of motion for the position and momentum operators, $\hat{X}_H(t)$ and $\hat{P}_H(t)$, for this system.

$$i\hbar \frac{d}{dt}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}_H(t)] \qquad \text{For } \hat{O} = \hat{O}_S \text{ time-indep.}$$

$$\hat{O}_H(t) := \hat{U}(t,t_0)^{-1}\hat{O}\,\hat{U}(t,t_0) \quad \& \quad \hat{H}_H(t) = \hat{U}(t,t_0)^{-1}\hat{H}\,\hat{U}(t,t_0) = \hat{H}$$

$$\Rightarrow [\hat{O}_H(t), \hat{H}_H(t)] = [\hat{U}(t,t_0)^{-1}\hat{O}\,\hat{U}(t,t_0), \hat{U}(t,t_0)^{-1}\hat{H}\,\hat{U}(t,t_0)]$$
$$= \hat{U}(t,t_0)^{-1}[\hat{O},\hat{H}]\,\hat{U}(t,t_0) \qquad ①$$

$$[\hat{X},\hat{H}] = [\hat{X}, \tfrac{1}{2}(\alpha\hat{P}^2 + \beta(\hat{X}\hat{P}+\hat{P}\hat{X}) + \gamma\hat{X}^2)]$$
$$= \frac{\alpha}{2}\underbrace{[\hat{X},\hat{P}^2]}_{2i\hbar\hat{P}} + \frac{\beta}{2}\underbrace{[\hat{X},\hat{X}\hat{P}+\hat{P}\hat{X}]}_{2i\hbar\hat{X}} + \frac{\gamma}{2}\underbrace{[\hat{X},\hat{X}^2]}_{\to 0}$$
$$= i\hbar(\alpha\hat{P} + \beta\hat{X}) \qquad ②$$

$$[\hat{P},\hat{H}] = [\hat{P}, \tfrac{1}{2}(\alpha\hat{P}^2 + \beta(\hat{X}\hat{P}+\hat{P}\hat{X}) + \gamma\hat{X}^2)]$$
$$= \frac{\alpha}{2}\underbrace{[\hat{P},\hat{P}^2]}_{\to 0} + \frac{\beta}{2}\underbrace{[\hat{P},\hat{X}\hat{P}+\hat{P}\hat{X}]}_{-2i\hbar\hat{P}} + \frac{\gamma}{2}\underbrace{[\hat{P},\hat{X}^2]}_{-2i\hbar\hat{X}}$$
$$= -i\hbar(\beta\hat{P} + \gamma\hat{X}) \qquad ③$$

$$①,②,③ \Rightarrow \quad [\hat{X}_H(t), \hat{H}_H] = i\hbar[\alpha\hat{P}_H(t) + \beta\hat{X}_H(t)] \quad \&$$
$$[\hat{P}_H(t), \hat{H}_H] = -i\hbar[\beta\hat{P}_H(t) + \gamma\hat{X}_H(t)]$$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\hat{X}_H(t) = \beta\hat{X}_H(t) + \alpha\hat{P}_H(t) \\[2ex] \dfrac{d}{dt}\hat{P}_H(t) = -\gamma\hat{X}_H(t) - \beta\hat{P}_H(t) \end{cases}$$

**3.b** (10 points) For the special case where $\beta = 0$, solve the equations you find in Problem 3.a to determine $\hat{X}_H(t)$ and $\hat{P}_H(t)$, if at some initial time $t_0$ we have $\hat{X}_H(t_0) = \hat{X}$ and $\hat{P}_H(t_0) = \hat{P}$.

$$\frac{d}{dt}\hat{X}_H(t) \overset{①}{=} \alpha \hat{P}_H(t)$$

$$\frac{d}{dt}\hat{P}_H(t) \overset{②}{=} -\gamma \hat{X}_H(t)$$

$$\Rightarrow \frac{d^2}{dt^2}\hat{X}_H(t) \overset{⓪}{=} -\alpha\gamma \hat{X}_H$$

If $\alpha = 0$: $\overset{①}{\Rightarrow}$ $\hat{X}_H(t) = \hat{X}_H(t_0) = \hat{X}$

② $\Rightarrow$ $\hat{P}_H(t) = -\gamma \hat{X} t + \hat{A}$

$\hat{P}_H(t_0) = \hat{P}$ $\Rightarrow$ $\hat{A} = \hat{P} + \gamma \hat{X} t_0$

$\Rightarrow$ $\hat{P}_H(t) = \hat{P} - \gamma(t - t_0)\hat{X}$

If $\gamma = 0$: $\overset{②}{\Rightarrow}$ $\hat{P}_H(t) = \hat{P}_H(t_0) = \hat{P}$

① $\hat{X}_H(t) = \alpha \hat{P} t + \hat{B}$

$\hat{X}_H(t_0) = \hat{X}$ $\Rightarrow$ $\hat{B} = \hat{X} - \alpha \hat{P} t_0$ $\Rightarrow$ $\hat{X}_H(t) = \hat{X} + \alpha(t - t_0)\hat{P}$

If $\alpha\gamma < 0$:

$\omega := \sqrt{|\alpha\gamma|}$

$\overset{⓪}{\Rightarrow}$ $\hat{X}_H(t) = \hat{A}_1 \cos[\omega(t - t_0)] + \hat{A}_2 \sin[\omega(t - t_0)]$

$\overset{①}{\Rightarrow}$ $\hat{P}_H(t) = \frac{\omega}{\alpha}\left(-\hat{A}_1 \sin[\omega(t - t_0)] + \hat{A}_2 \cos[\omega(t - t_0)]\right)$

$\hat{X}_H(t_0) \overset{③}{=} \hat{X}$ & $\hat{P}_H(t_0) \overset{④}{=} \hat{P}$ $\Rightarrow$ $\hat{A}_1 = \hat{X}$ & $\hat{A}_2 = \frac{\alpha}{\omega}\hat{P}$

$$\Rightarrow \begin{cases} \hat{X}_H(t) = \cos[\omega(t - t_0)]\hat{X} + \frac{\alpha}{\omega}\sin[\omega(t - t_0)]\hat{P} \\ \hat{P}_H(t) = -\frac{\omega}{\alpha}\sin[\omega(t - t_0)]\hat{X} + \cos[\omega(t - t_0)]\hat{P} \end{cases}$$

If $\alpha\gamma > 0$:

$k := \sqrt{\alpha\gamma}$

$\overset{⓪}{\Rightarrow}$ $\hat{X}_H(t) = \hat{B}_1 \cosh[k(t - t_0)] + \hat{B}_2 \sinh[k(t - t_0)]$

$\overset{①}{\Rightarrow}$ $\hat{P}_H(t) = \frac{k}{\alpha}\left(\hat{B}_1 \sinh[k(t - t_0)] + \hat{B}_2 \cosh[k(t - t_0)]\right)$

$$\left.\begin{array}{l} ③ \Rightarrow \hat{B}_1 = \hat{X} \\ ④ \Rightarrow \hat{B}_2 = \frac{\alpha}{k}\hat{P} \end{array}\right\} \Rightarrow \begin{cases} \hat{X}_H(t) = \cosh[k(t - t_0)]\hat{X} + \frac{\alpha}{k}\sinh[k(t - t_0)]\hat{P} \\ \hat{P}_H(t) = \frac{k}{\alpha}\sinh[k(t - t_0)]\hat{X} + \cosh[k(t - t_0)]\hat{P} \end{cases}$$

**Problem 4** Let $\mathscr{H}$ be a two-dimensional Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle\}$, $\hat{A} := |1\rangle\langle 2| + |2\rangle\langle 1|$, and

$$\hat{H} := E\left(|1\rangle\langle 1| - i|1\rangle\langle 2| + i|2\rangle\langle 1| + |2\rangle\langle 2|\right),$$

where $E$ is a real and positive constant. A quantum system, that is described by the Hilbert space $\mathscr{H}$ and the Hamiltonian operator $\hat{H}$, is in the state determined by the state vector $|1\rangle + |2\rangle$ at time $t = 0$.

**4.a** (5 points) Find the eigenvalues and eigenvectors of $\hat{H}$.

$$\mathbb{H} = [\langle i|\hat{H}|j\rangle] = \begin{bmatrix} E & -iE \\ iE & E \end{bmatrix} \qquad \mathbb{H}\,\mathcal{H} = \lambda\,\mathcal{H} \Rightarrow$$

$$\begin{bmatrix} E-\lambda & -iE \\ iE & E-\lambda \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (E-\lambda)^2 - E^2 = 0 \Rightarrow \lambda - E = \pm E \Rightarrow \lambda = \begin{cases} \lambda_1 := 0 \\ \lambda_2 := 2E \end{cases}$$

For $\lambda = \lambda_1 = 0$: $\begin{bmatrix} E & -iE \\ iE & E \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow E(\alpha_1 - i\beta_1) = 0 \Rightarrow \alpha_1 = i\beta_1 \Rightarrow \mathcal{H}_1 = \beta_1 \begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

choose $\alpha_2 = \frac{1}{\sqrt{2}}$

For $\lambda = \lambda_2 = 2E$

$$\begin{bmatrix} -E & -iE \\ iE & -E \end{bmatrix}\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow E(\alpha_2 + i\beta_2) = 0$$

$$\Rightarrow \beta_2 = i\alpha_2 \Rightarrow \mathcal{H}_2 = \alpha_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

choose $\alpha_2 = \frac{1}{\sqrt{2}}$

$$\mathcal{H}_i = \begin{bmatrix} \langle 1|\psi_i\rangle \\ \langle 2|\psi_i\rangle \end{bmatrix} \Rightarrow$$

Eigenvectors of $\hat{H}$ are

$|\psi_1\rangle = \frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle)$ with eigenvalue $\lambda = 0$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$ " " $\lambda = 2E$

**4.b** (15 points) Show that the time-evolution operator $\hat{U}(t, t_0)$ for this system has the form

$$\hat{U}(t,t_0) = f(t,t_0)\hat{I} + g(t,t_0)\hat{B},$$

where $f(t,t_0)$ and $g(t,t_0)$ are complex-valued functions, $\hat{I}$ is the identity operator acting in $\mathscr{H}$, and $\hat{B}$ is a Hermitian operator. Determine the explicit expression for $f(t,t_0)$, $g(t,t_0)$, and $\hat{B}$.

$$\hat{U}(t,t_0) = e^{-\frac{i(t-t_0)}{\hbar}\hat{H}} \overset{①}{=} \sum_{i=1}^{2} e^{-\frac{i(t-t_0)\lambda_i}{\hbar}} |\psi_i\rangle\langle\psi_i|$$

$$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(i|1\rangle + |2\rangle)(-i\langle 1| + \langle 2|)$$

$$= \frac{1}{2}(\underbrace{|1\rangle\langle 1| + |2\rangle\langle 2|}_{\hat{I}} + i|1\rangle\langle 2| - i|2\rangle\langle 1|)$$

$$\overset{②}{=} \frac{1}{2}[\hat{I} + i(|1\rangle\langle 2| - |2\rangle\langle 1|)]$$

$$|\psi_2\rangle\langle\psi_2| = \frac{1}{2}(|1\rangle + i|2\rangle)(\langle 1| - i\langle 2|)$$

$$= \frac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2| - i|1\rangle\langle 2| + i|2\rangle\langle 1|)$$

$$\overset{③}{=} \frac{1}{2}[\hat{I} - i(|1\rangle\langle 2| - |2\rangle\langle 1|)]$$

$$①\text{-}③ \Rightarrow \hat{U}(t,t_0) = |\psi_1\rangle\langle\psi_1| + e^{-\frac{2i(t-t_0)E}{\hbar}} |\psi_2\rangle\langle\psi_2|$$

$$= \frac{1}{2}[\hat{I} + i(|1\rangle\langle 2| - |2\rangle\langle 1|)] + \frac{e^{-\frac{2i(t-t_0)E}{\hbar}}}{2}[\hat{I} - i(|1\rangle\langle 2| - |2\rangle\langle 1|)]$$

$$= \underbrace{\frac{1}{2}\left(1 + e^{-\frac{2i(t-t_0)E}{\hbar}}\right)}_{f(t,t_0)} \hat{I} + \underbrace{\frac{1}{2}\left(1 - e^{-\frac{2i(t-t_0)E}{\hbar}}\right)}_{g(t,t_0)} \hat{B}$$

$$\hat{B} := i(|1\rangle\langle 2| - |2\rangle\langle 1|).$$

**4.c** (10 points) Find the expectation value of $\hat{A}$ at time $t \geq 0$.

$$\langle \hat{A} \rangle_{\psi(t)} = \frac{\langle \psi(t) | \hat{A} | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} = \frac{\langle \psi(t) | \hat{A} | \psi(t) \rangle}{\langle \psi(0) | \psi(0) \rangle}, \quad t_0 = 0$$

$$|\psi_0\rangle := |1\rangle + |2\rangle \qquad \langle \psi(0) | \psi(0) \rangle = \langle \psi_0 | \psi_0 \rangle \doteq 2 \quad ①$$

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi_0\rangle = [f(t,t_0)\hat{I} + g(t,t_0)\hat{B}]|\psi_0\rangle$$

$$\hat{B}|\psi_0\rangle = i(|1\rangle\langle 2| - |2\rangle\langle 1|)(|1\rangle + |2\rangle) = i(|1\rangle - |2\rangle)$$

$$\Rightarrow |\psi(t)\rangle = f(t,t_0)(|1\rangle + |2\rangle) + ig(t,t_0)(|1\rangle - |2\rangle)$$

$$= \underbrace{[f(t,t_0) + ig(t,t_0)]}_{h_+(t)}|1\rangle + \underbrace{[f(t,t_0) - ig(t,t_0)]}_{h_-(t)}|2\rangle$$

$t_0 = 0$

$$h_\pm(t) := \frac{1}{2}\left[(1 \pm i) + e^{-\frac{2iEt}{\hbar}}(1 \mp i)\right]$$

$$\hat{A}|\psi(t)\rangle = (|1\rangle\langle 2| + |2\rangle\langle 1|)[h_+(t)|1\rangle + h_-(t)|2\rangle]$$

$$= h_+(t)|2\rangle + h_-(t)|1\rangle$$

$$\langle \psi(t) | \hat{A} | \psi(t) \rangle = (h_+(t)^* \langle 1| + h_-(t)^* \langle 2|)(h_+(t)|2\rangle + h_-(t)|1\rangle)$$

$$= h_+^*(t) h_-(t) + h_-^*(t) h_+(t)$$

$$= 2\,\mathrm{Re}[h_+(t)^* h_-(t)]$$

$$= \frac{1}{2}\mathrm{Re}\left[\left\{(1-i) + e^{\frac{2iEt}{\hbar}}(1+i)\right\}\left\{(1-i) + e^{-\frac{2iEt}{\hbar}}(1+i)\right\}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[(1-i)^2 + (1+i)^2 + 2\left(e^{\frac{2iEt}{\hbar}} + e^{-\frac{2iEt}{\hbar}}\right)\right]$$

$$= 2\cos\left(\frac{2Et}{\hbar}\right) \quad ②$$

$$① \,\&\, ② \Rightarrow \langle \hat{A} \rangle_{\psi(t)} = \cos\left(\frac{2Et}{\hbar}\right)$$

**4.d** (10 points) Calculate the probability of finding the value 1 for the measurement of $\hat{A}$ at time $t \geq 0$.

First we solve the eigenvalue problem for $\hat{A}$:

$$\mathbb{A} = [\langle i|\hat{A}|j\rangle] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues & an orthonormal set of eigenvectors are

$$\alpha_1 := 1, \quad a_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow |\alpha_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$\alpha_2 := -1, \quad a_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow |\alpha_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

Probability of finding $\alpha = \alpha_1 = 1$ is

$$\mathcal{P}_{\alpha_1}(\lambda_{\psi(t)}) = \frac{|\langle\alpha_1|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{|\langle\alpha_1|\psi(t)\rangle|^2}{\underbrace{\langle\psi_0|\psi_0\rangle}_{\to 2}}$$

$$\langle\alpha_1|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\langle 1| + \langle 2|)\left[h_+(t)\,|1\rangle + h_-(t)\,|2\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[h_+(t) + h_-(t)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(2 + 2e^{-2iEt/\hbar}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(1 + e^{-2iEt/\hbar}\right)$$

$$|\langle\alpha_1|\psi(t)\rangle|^2 = \frac{1}{2}\left(1 + e^{-2iEt/\hbar}\right)\left(1 + e^{2iEt/\hbar}\right)$$

$$= \frac{1}{2}\left[2 + 2\cos\left(\frac{2Et}{\hbar}\right)\right]$$

$$= 1 + \cos\left(\frac{2Et}{\hbar}\right)$$

$$\Rightarrow \mathcal{P}_{\alpha_1}(\lambda_{\psi(t)}) = \frac{1}{2}\left[1 + \cos\left(\frac{2Et}{\hbar}\right)\right]$$

**4.e** (10 points) Find the shortest time it takes for the state vector $|1\rangle + |2\rangle$ to evolve into a state vector that is orthogonal to $|1\rangle + |2\rangle$.

If $|\psi_0\rangle$ evolves into $|\psi(t)\rangle$ that is orthogonal to $|\psi_0\rangle = |1\rangle + |2\rangle = \sqrt{2}\,|\alpha_1\rangle$

Then at time $t$, the probability to find the value $\alpha = \alpha_1$ in measuring $\hat{A}$ must be zero.

$\Updownarrow$ by part 4.d

$$\mathcal{P}_{\alpha_1}(\lambda_{\psi(t)}) = 0 \iff \cos\left(\frac{2Et}{\hbar}\right) = -1$$

$$\Rightarrow \quad \frac{2Et}{\hbar} = (2n+1)\pi \qquad n \in \mathbb{Z}$$

$$\Rightarrow \quad t = \frac{(2n+1)\pi\hbar}{2E}$$

So the shortest time is $t_{min} = \dfrac{\pi\hbar}{2E}$.