In the following problems $\hat{X}$ and $\hat{P}$ denote the standard position and momentum operators acting in $L^2(\mathbb{R})$.

1 (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued analytic function, i.e., it has a convergent Maclaurin series with an infinite radius of convergence, so that $f(x) = \sum_{n=1}^{\infty} a_n x^n$ for all $x \in \mathbb{R}$ and $a_n := f^{(n)}(0)/n!$, and $f'(x)$ denotes the first derivative of f. Show that the following identities hold.

a) $[\hat{X}, f(\hat{P})] = i\hbar f'(\hat{P})$.

b) $[\hat{P}, f(\hat{X})] = -i\hbar f'(\hat{X})$.

2 (10 points) Let α and κ be real parameters with the dimension of length and momentum, respectively, and $\psi \in L^2(\mathbb{R})$. Show that the following identities hold.

a) $\langle x|e^{-i\alpha\hat{P}/\hbar}|\psi\rangle = \langle x - \alpha|\psi\rangle$.

b) $\langle p|e^{-i\kappa\hat{X}/\hbar}|\psi\rangle = \langle p + \kappa|\psi\rangle$.

3 (15 points) A particle of mass m is described by the Hilbert space $L^2(\mathbb{R})$ and the Hamiltonian operator $\hat{H} = \hat{P}^4/2m^3c^2$, where c is the speed of light. Find the spectrum, degeneracy of the energy levels, and the position wave functions for the (generalized) eigenvectors of $\hat{H}$.

4 Consider a free particle of mass m that moves in a straight line. Suppose that at time $t = 0$ the particle in a state given by the position wave function:

$$\psi_0(x) = e^{ip_0x/\hbar}e^{-(x-a)^2/4\sigma^2},$$

where p_0, a, σ are real parameters, and $\sigma > 0$. Calculate the following quantities:

4.a (10 points) The position wave function at times $t > 0$.

4.b (10 points) Expectation value of the position $\hat{X}$ of this particle at times $t > 0$.

4.c (10 points) Uncertainty in the position of this particle at times $t > 0$.

4.d (5 points) The momentum wave function at times $t > 0$.

4.e (5 points) Expectation value of the momentum $\hat{P}$ of this particle at times $t > 0$.

4.f (5 points) Uncertainty in the momentum of this particle at times $t > 0$.

4.g (10 points) Explain how the expectation value of the momentum of the particle compares with the time-derivative of the expectation value of its position.