Phys 402/OEPE 542: Midterm Exam 1

March 20, 2018

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 Let $\hat{X}_j$ and $\hat{P}_j$ be respectively the standard position and momentum operators acting in the Hilbert space $L^2(\mathbb{R}^N)$, and $\mathcal{T}: L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the time-reversal operator defined by $(\mathcal{T}\psi)(\vec{x}) = \psi(\vec{x})^*$. Show that the following relations hold.

1.a (5 points) $[\mathcal{T}, \hat{X}_j] = 0$.

$$([\mathcal{T}, \hat{X}_j]\psi)(\vec{x}) = (\mathcal{T}\hat{X}_j\psi)(\vec{x}) - (\hat{X}_j\mathcal{T}\psi)(\vec{x})$$
$$= (\hat{X}_j\psi)(\vec{x})^* - \hat{X}_j\psi(\vec{x})^*$$
$$= x_j\psi(\vec{x})^* - x_j\psi(\vec{x})^* = 0 \quad \Rightarrow \quad [\mathcal{T}, \hat{X}_j] = \hat{0}$$

1.b (5 points) $\{\mathcal{T}, \hat{P}_j\} = 0$.

$$(\{\mathcal{T}, \hat{P}_j\}\psi)(\vec{x}) = ((\mathcal{T}\hat{P}_j)\psi)(\vec{x}) + ((\hat{P}_j\mathcal{T})(\psi))(\vec{x})$$
$$= (\hat{P}_j\psi)(\vec{x})^* + (\hat{P}_j\psi^*)(\vec{x})$$
$$= \left[-i\frac{\partial}{\partial x_j}\psi(\vec{x})\right]^* + -i\frac{\partial}{\partial x_j}\left[\psi(\vec{x})^*\right]$$
$$= i\left[\frac{\partial}{\partial x_j}\psi(\vec{x})\right]^* - i\left[\frac{\partial}{\partial x_j}\psi(\vec{x})\right]^*$$
$$= 0$$
$$\Rightarrow \quad \{\mathcal{T}, \hat{P}_j\} = \hat{0}.$$

Problem 2 Let $\hat{H}_\varepsilon = \hbar\omega\{\hat{a}, \hat{a}^\dagger\} + \varepsilon(\hat{a}^{\dagger 2} + \hat{a}^2)$ where $\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P}$, and ε, ω, and m are positive real parameters of the dimension of energy, frequency, and mass, respectively.

2.a (10 points) Find all values of ϵ such that $\hat{H}_\varepsilon$ describes a simple harmonic oscillator.

$$\hat{a}^2 + \hat{a}^{+2} = \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P}\right)^2 + \left(\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{i\hat{P}}{\sqrt{2\hbar m\omega}}\right)^2$$

$$= 2\left[\frac{m\omega}{2\hbar}\hat{X}^2 - \frac{1}{2\hbar m\omega}\hat{P}^2\right]$$

$$\hbar\omega\{\hat{a}, \hat{a}^+\} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2$$

$$\Rightarrow \quad \hat{H}_\epsilon = \frac{\hat{P}}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + \frac{2\epsilon}{\hbar\omega}\left(-\frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\hat{X}^2\right)$$

$$= \left(1 - \frac{2\epsilon}{\hbar\omega}\right)\frac{\hat{P}^2}{2m} + \left(1 + \frac{2\epsilon}{\hbar\omega}\right)\frac{1}{2}m\omega^2\hat{X}^2$$

This defines a SHO for $\left(1 - \frac{2\epsilon}{\hbar\omega} > 0 \;\&\; 1 + \frac{2\epsilon}{\hbar\omega} > 0\right)$

$$\Leftrightarrow \left(\epsilon < \frac{\hbar\omega}{2} \;\&\; \epsilon > -\frac{\hbar\omega}{2}\right)$$

$$\Leftrightarrow \quad \boxed{|\epsilon| < \frac{\hbar\omega}{2}}$$

2.b (5 points) Find the ground state energy of $\hat{H}_\varepsilon$ for the values of ε you find in part a of this problem.

$$\left(1 - \frac{2\epsilon}{\hbar\omega}\right)\frac{1}{2m} = \frac{1}{2M} \Rightarrow M = \frac{m}{1 - \frac{2\epsilon}{\hbar\omega}}$$

$$\left(1 + \frac{2\epsilon}{\hbar\omega}\right)\frac{1}{2}m\omega^2 = \frac{1}{2}M\Omega^2$$

$$\Rightarrow \quad \Omega = \sqrt{\left(1 + \frac{2\epsilon}{\hbar\omega}\right)\frac{m}{M}}\,\omega = \sqrt{\left(1 + \frac{2\epsilon}{\hbar\omega}\right)\left(1 - \frac{2\epsilon}{\hbar\omega}\right)}\,\omega$$

$$= \sqrt{1 - \left(\frac{2\epsilon}{\hbar\omega}\right)^2}\,\omega$$

$$\Rightarrow \quad \hat{H}_\epsilon = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\Omega^2\hat{X}^2$$

Ground state energy $\quad E_0 = \frac{\hbar\Omega}{2} = \frac{\hbar\omega}{2}\sqrt{1 - \left(\frac{2\epsilon}{\hbar\omega}\right)^2}$

$$\Rightarrow \quad \boxed{E_0 = \frac{1}{2}\sqrt{(\hbar\omega)^2 - 4\epsilon^2}}$$

Problem 3 Let $\hat{H} = \hbar\omega\{\hat{a}, \hat{a}^\dagger\}$ be the Hamiltonian for simple harmonic oscillator of mass m and angular frequency ω, and $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$, where α is a real number belonging to the interval $[0, 2\pi)$, and $|0\rangle$ and $|1\rangle$ are the state vectors for the ground state and first excited state of the oscillator. Calculate the following quantities.

3.a (10 points) The uncertainty in the position of this oscillator when it is in the state described by $|\psi\rangle$.

$\langle\psi|\psi\rangle = 1, \quad \hat{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\hat{a} + \hat{a}^\dagger)$

$\hat{a}|0\rangle = 0, \quad \hat{a}^\dagger|0\rangle = |1\rangle, \quad \hat{a}|1\rangle = |0\rangle, \quad \hat{a}^\dagger|1\rangle = \sqrt{2}|2\rangle$

$$\Rightarrow \quad \hat{x}|\psi\rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\hat{a} + \hat{a}^\dagger)\frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}(e^{i\alpha}|0\rangle + |1\rangle + e^{i\alpha}\sqrt{2}|2\rangle)$$

$$\Rightarrow \quad \langle\psi|\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}\left(\langle 0| + e^{-i\alpha}\langle 1|\right)\left(e^{i\alpha}|0\rangle + |1\rangle + e^{i\alpha}\sqrt{2}|2\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}(e^{i\alpha} + e^{-i\alpha}) = \sqrt{\frac{\hbar}{2m\omega}}\cos\alpha$$

$$\langle\psi|\hat{x}^2|\psi\rangle = \|\hat{x}|\psi\rangle\|^2$$

$$= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)(1 + 1 + 2)$$

$$= \frac{\hbar}{m\omega}$$

$$\Rightarrow \quad \Delta_\psi \hat{x} = \sqrt{\langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|\hat{x}|\psi\rangle^2}$$

$$= \sqrt{\frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\cos^2\alpha} = \sqrt{\frac{\hbar}{2m\omega}\underbrace{\left(2 - \frac{1+\cos 2\alpha}{2}\right)}_{\frac{3}{2} - \frac{1}{2}\cos 2\alpha}}$$

$$\Rightarrow \quad \boxed{\Delta_\psi \hat{x} = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}(3 - \cos 2\alpha)}}$$

3.b (5 points) The uncertainty in the momentum of this oscillator when it is in the state described by $|\psi\rangle$.

$$\hat{P} = \frac{1}{2i}\sqrt{2\hbar m\omega}\,(\hat{a}-\hat{a}^{+})$$

$$\hat{P}|\psi\rangle = \frac{1}{2i}\sqrt{\hbar m\omega}\,(\hat{a}-\hat{a}^{+})(|0\rangle + e^{i\alpha}|1\rangle)$$

$$= \frac{1}{2i}\sqrt{\hbar m\omega}\,(e^{i\alpha}|0\rangle - |1\rangle - e^{i\alpha}\sqrt{2}|2\rangle)$$

$$\Rightarrow \langle\psi|\hat{P}|\psi\rangle = \frac{1}{2i}\sqrt{\frac{\hbar m\omega}{2}}\left(\langle 0| + e^{-i\alpha}\langle 1|\right)\left(e^{i\alpha}|0\rangle - |1\rangle - e^{i\alpha}\sqrt{2}|2\rangle\right)$$

$$= \frac{1}{2i}\sqrt{\frac{\hbar m\omega}{2}}\,(e^{i\alpha} - e^{-i\alpha})$$

$$= \sqrt{\frac{\hbar m\omega}{2}}\,\sin\alpha$$

$$\langle\psi|\hat{P}^2|\psi\rangle = \|\hat{P}|\psi\rangle\|^2 = \frac{1}{4}(\hbar m\omega)(1+1+2)$$

$$= \hbar m\omega$$

$$\Delta_\psi \hat{P} = \sqrt{\langle\psi|\hat{P}^2|\psi\rangle - \langle\psi|\hat{P}|\psi\rangle^2}$$

$$= \sqrt{\hbar m\omega - \frac{\hbar m\omega}{2}\sin^2\alpha} = \sqrt{\frac{\hbar m\omega}{2}\underbrace{(2 - \underbrace{\sin^2\alpha}_{\frac{1-\cos 2\alpha}{2}})}_{\frac{3}{2}+\frac{\cos 2\alpha}{2}}}$$

$$\Rightarrow \boxed{\Delta_\psi \hat{P} = \frac{1}{2}\sqrt{\hbar m\omega(3+\cos 2\alpha)}}$$

3.c (5 points) Find the values of α for which the product of the uncertainties you compute in parts a and b of this problem is largest. What is the largest value of the product of these uncertainties?

$$\Delta \hat{x}_t \, \Delta \hat{p}_t = \frac{\hbar}{4}\sqrt{(3-\cos 2\alpha)(3+\cos 2\alpha)}$$

$$= \frac{\hbar}{4}\sqrt{9-\cos^2 2\alpha}$$

$$= \frac{3\hbar}{4}\sqrt{1-\frac{\cos^2 2\alpha}{9}}$$

The largest value is attained for

$$\cos 2\alpha = 0 \Rightarrow \quad 2\alpha = \frac{\pi}{2} + \pi n \quad n \in \mathbb{Z}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4}(2n+1) \quad n \in \mathbb{Z}$$

For $\alpha \in [0, 2\pi) \quad \Rightarrow \quad \boxed{\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

The largest value is $\frac{3\hbar}{4}$.

Problem 4 (15 points) Consider a delta function potential of the form $v(x) = -\lambda\,\delta(x)$ in one dimension for some $\lambda > 0$. The quantum system defined by the Hamiltonian operator $\hat{H} = \frac{P^2}{2m} + v(\hat{X})$ has a single bound state with a negative energy. Find the energy of this state and a corresponding position wave function. You do not need to normalize this wave function.

$$-\frac{\hbar^2}{2m}\psi'' - \lambda\,\delta(x)\,\psi = E\,\psi$$

For ground state $E_0 < 0 \Rightarrow E_0 = -|E_0|$

$$\Rightarrow \psi''_{E_0}(x) + \frac{2m\lambda}{\hbar^2}\delta(x)\,\psi_{E_0}(x) = \frac{2m|E_0|}{\hbar^2}\psi(x)$$

Let $\xi := \frac{2m\lambda}{\hbar^2}$, $k := \frac{\sqrt{2m|E_0|}}{\hbar}$

$$\Rightarrow \psi''_{E_0}(x) + \xi\,\delta(x)\,\psi_{E_0}(x) = k^2\psi(x)$$

For $x \neq 0$: $\psi''_{E_0} - k^2\psi_{E_0} = 0$

For $x < 0$ $\hookrightarrow$ $\psi_{E_0}(x) = Ae^{kx}$ So that $\psi(x) \to 0$ for $x \to -\infty$

For $x > 0$ $\psi_{E_0}(x) = Be^{-kx}$ " " $\psi(x) \to 0$ for $x \to +\infty$

$\psi_{E_0}(0^-) = \psi_{E_0}(0^+) \hookrightarrow \boxed{A = B}$

Also $\underbrace{\int_{-\epsilon}^{\epsilon} dx\,[\psi''_{E_0}(x) + \xi\,\delta(x)\,\psi_{E_0}(x)]}$ $= k^2\int_{-\epsilon}^{\epsilon}\psi_{E_0}(x)\,dx$

$$\psi'_{E_0}(\epsilon) - \psi'_{E_0}(-\epsilon) + \xi\,\psi_{E_0}(0) = k^2\int_{-\epsilon}^{\epsilon}\psi_{E_0}(x)\,dx$$

as $\epsilon \to 0^+$, we find $\boxed{\psi'_{E_0}(0^+) - \psi'_{E_0}(0^-) + \xi\,\psi_{E_0}(0) = 0}$

$\psi'_{E_0}(0^+) = -kB = -kA$ $\hookrightarrow$ $-kA - kA + \xi A = 0$

$\psi'_{E_0}(0^-) = kA$ $\Downarrow$

$\psi_{E_0}(0) = A$ $k = \frac{\xi}{2} = \frac{m\lambda}{\hbar^2}$

$$\Rightarrow \boxed{E_0 = -|E_0| = -\frac{(\hbar k)^2}{2m} = -\frac{1}{2m}\left(\frac{m\lambda}{\hbar}\right)^2 = -\frac{m\lambda^2}{2\hbar^2}}$$

$\Downarrow$

$$\psi_{E_0}(x) = \begin{cases} A e^{kx} & x < 0 \\ A e^{-kx} & x \leq 0 \end{cases}$$

with $k = \frac{m\lambda}{\hbar^2}$

Problem 5 Consider a quantum system with Hilbert space $L^2(\mathbb{R})$ and a standard Hamiltonian $\hat{H} = \frac{1}{2}(\hat{P}_x^2+\hat{P}_y^2)+v(\hat{X},\hat{Y})$, where $v(x,y) = -\lambda\,\delta(x)+\frac{1}{2}m\omega^2 y^2$, and m, ω, λ are positive real parameters.

5.a (10 points) Find the ground state energy of the system.
Hint: You may use your response to Problem 4.

Try separation of variables:

$$\psi_E(x,y) = X(x)\,Y(y) \quad \Rightarrow$$

$$\left[-\frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2) - \lambda\,\delta(x) + \frac{1}{2}m\omega^2 y^2\right]\psi_E = E\psi_E$$

$$\Downarrow$$

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{X''}{X} - \lambda\,\delta(x)\right]}_{\varepsilon} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{Y''}{Y} + \frac{1}{2}m\omega^2 y^2\right]}_{\varepsilon'} = E$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\,\delta(x)\right]X(x) = \varepsilon\, X(x)$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2\right]Y(x) = \varepsilon' Y(x)$$

$$E = \varepsilon + \varepsilon'$$

ground state of $\frac{1}{2}m\omega^2 y^2$

ground state : $E_0 = \varepsilon_0 + \varepsilon'_0$

ground state of $-\lambda\,\delta(x)$

$$\varepsilon_0 = -\frac{m\lambda^2}{2\hbar^2} \quad \text{by problem 4}$$

$$\varepsilon'_0 = \frac{\hbar\omega}{2}$$

So $\boxed{E_0 = \frac{\hbar\omega}{2} - \frac{m\lambda^2}{2\hbar^2}}$

5.b (10 points) Show that this system has finitely many bound states and determine their number.

$-\lambda\delta(x)$ has a single negative eigenvalue and a continuous spectrum given by $[0,\infty)$

So the spectrum of $\hat{H}$ is

$$\mathrm{Spec}(\hat{H}) = \left\{ \hbar\omega(n+\tfrac{1}{2}) - \frac{m\lambda^2}{2\hbar^2} \;\middle|\; n\in\{0,\dots,n_{max}\} \right\}$$

$$\cup \left\{ \frac{\hbar\omega}{2} + k^2 \;\middle|\; k\in[0,\infty) \right\}$$

$$\hbar\omega(n_{max}+\tfrac{1}{2}) - \frac{m\lambda^2}{2\hbar^2} < \frac{\hbar\omega}{2}$$

$$\Leftrightarrow \quad n_{max} < \frac{m\lambda^2}{2m\omega\hbar^3}$$

So n_{max} is the largest integer smaller than $\frac{m\lambda^2}{2m\omega\hbar^2}$

i.e. n_{max} = integer part of $\frac{m\lambda^2}{2m\omega\hbar^3}$ if this number is not an integer. Otherwise it is $\frac{m\lambda^2}{2m\omega\hbar^3} - 1$.

Problem 6 Let a and N_a be a positive real numbers and $\hat{U}_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(\hat{U}_a\psi)(x) := N_a\psi(ax)$, where $\hat{X}$ and $\hat{P}$ be the standard position and momentum operator acting in $L^2(\mathbb{R})$.

6.a (5 points) Find N_a so that $\hat{U}_a$ is a unitary operator.

$$\|\hat{U}_a\psi\|^2 = \int_{-\infty}^{\infty} dx\, |N_a|^2\, \psi(ax)^*\, \psi(ax)$$

let $y := ax \Rightarrow dy = a\,dx$

$$\Rightarrow \|\hat{U}_a\psi\|^2 = \int_{-\infty}^{\infty} \frac{dy}{a} N_a^2\, \psi(y)^*\, \psi(y) = \frac{N_a^2}{a}\|\psi\|^2$$

$$\Rightarrow \boxed{N = \sqrt{a}}$$

$$\Rightarrow \boxed{(U_a\psi)(x) = \sqrt{a}\,\psi(aX)} \Rightarrow \boxed{\langle x|\hat{U}_a|\psi\rangle = \sqrt{a}\,\langle ax|\psi\rangle}$$

6.b (5 points) Find a positive real number b such that $U_b = U_a^{-1}$.

let $\phi(x) = (\hat{U}_a\psi)(x) \Rightarrow (\hat{U}_a^{-1}\phi)(x) = \psi(x)$

$= \sqrt{a}\,\psi(ax)$

$\Downarrow$

$y = ax \qquad \phi\left(\frac{y}{a}\right) = \sqrt{a}\,\psi(y) \Rightarrow \psi(y) = \frac{1}{\sqrt{a}}\phi\left(\frac{y}{a}\right)$

$$\boxed{(U^{-1}\phi)(x) = \frac{1}{\sqrt{a}}\phi\left(\frac{x}{a}\right)}$$

$= (U_b\phi)(x)$

$= \sqrt{b}\,\phi(bx)$

$\Downarrow$

$$\boxed{b = \frac{1}{a}}$$

$$\Rightarrow \boxed{\hat{U}_a^{-1} = \hat{U}_{\frac{1}{a}}}$$

6.c (5 points) Calculate $\hat{U}_a \hat{X} \hat{U}_a^{-1}$ for N_a you find in part a of this problem, i.e., express $\hat{U}_a \hat{X} \hat{U}_a^{-1}$ in terms of $\hat{X}$ and $\hat{P}$.

$$\langle x| \hat{U}_a \hat{X} \hat{U}_a^{-1} |\psi\rangle = \sqrt{a} \langle ax| \hat{X} \hat{U}_{\frac{1}{a}} |\psi\rangle$$

$$= \sqrt{a} \quad a x \quad \langle ax| \hat{U}_{\frac{1}{a}} |\psi\rangle$$

$$= \sqrt{a} \quad a x \quad \sqrt{\tfrac{1}{a}} \psi\left(\tfrac{1}{a}(ax)\right)$$

$$= a x \, \psi(x)$$

$$= \langle x| a \hat{X} |\psi\rangle$$

$$\Rightarrow \boxed{\hat{U}_a \hat{X} \hat{U}_a^{-1} = a \hat{X}}$$

6.d (5 points) Calculate $\hat{U}_a \hat{P} \hat{U}_a^{-1}$ for N_a you find in part a of this problem, i.e., express $\hat{U}_a \hat{P} \hat{U}_a^{-1}$ in terms of $\hat{X}$ and $\hat{P}$.

$$\langle x| \hat{U}_a \hat{P} \hat{U}_a^{-1} |\psi\rangle = \sqrt{a} \langle ax| \hat{P} \hat{U}_a^{-1} |\psi\rangle$$

$$= \sqrt{a} \quad -i\hbar \frac{d}{d(ax)} \underbrace{\langle ax| \hat{U}_{\frac{1}{a}} |\psi\rangle}_{\frac{1}{\sqrt{a}} \underbrace{\langle \frac{1}{a}(ax)|\psi\rangle}_{\psi(x)}}$$

$$= -\frac{i\hbar}{a} \frac{d}{dx} \psi(x)$$

$$= \frac{1}{a} \langle x| \hat{P} |\psi\rangle$$

$$\Rightarrow \boxed{\hat{U}_a \hat{P} \hat{U}_a^{-1} = \frac{1}{a} \hat{P}}$$