Solutions

Phys 402: Midterm Exam 1

March 15, 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 Let N be a positive integer, and for each $j \in \{1, 2, \cdots, N\}$, $\hat{X}_j$ and $\hat{P}_j$ be respectively the standard position and momentum operators acting in the Hilbert space $L^2(\mathbb{R}^N)$, i.e., $(\hat{X}_j\psi)(\vec{x}) = x_j\psi(\vec{x})$ and $(\hat{P}_j\psi)(\vec{x}) = -i\hbar\frac{\partial}{\partial x_j}\psi(\vec{x})$, where $x_1, x_2, \cdots, x_N$ are the Cartesian coordinates of $\vec{x}$.

1.a (5 points) Show that $\hat{P}_j$ is a symmetric operator, i.e., if ψ and ϕ belong to the domain of $\hat{P}_j$, we have $\langle\phi|\hat{P}_j\psi\rangle = \langle\hat{P}_j\phi|\psi\rangle$.

$$\langle\phi|\hat{P}_j|\psi\rangle = \int_{\mathbb{R}^N} d^N x\, \underbrace{\phi(\vec{x})^* \left[-i\hbar\frac{\partial}{\partial x_j}\psi(x)\right]}_{-i\hbar\left[\frac{\partial}{\partial x_j}[\phi(\vec{x})^*\psi(x)] - \left(\frac{\partial}{\partial x_j}\phi(\vec{x})^*\right)\psi(x)\right]}$$

$$\int_{-\infty}^{\infty} dx_j \frac{\partial}{\partial x_j}[\phi(x)^*\psi(x)] = \phi(\vec{x})^*\psi(x)\Big|_{x_j=-\infty}^{x_j=+\infty} = 0$$

Because $\langle\psi|\psi\rangle < \infty$

$$\Downarrow$$

$$\begin{cases} \lim_{|\vec{x}|\to\infty}\psi(\vec{x}) = 0 \\ \lim_{|\vec{x}|\to\infty}\phi(\vec{x}) = 0 \end{cases}$$

$$\Rightarrow \int_{\mathbb{R}^N} d^N x \frac{\partial}{\partial x_j}[\phi(\vec{x})^*\psi(x)] = 0$$

$$\langle\phi|\hat{P}_j|\psi\rangle = \int_{\mathbb{R}^N} d^N x\, \underbrace{i\hbar\left(\frac{\partial}{\partial x_j}\phi(\vec{x})^*\right)}_{\left[-i\hbar\frac{\partial}{\partial x_j}\phi(\vec{x})\right]^*}\psi(\vec{x})$$

$$= \langle\hat{P}_j\phi|\psi\rangle \quad \blacksquare$$

1.b (5 points) Let $f : \mathbb{R}^N \to \mathbb{R}$ be a differentiable function, and $\widehat{\vec{X}} := (\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_N)$. Derive an expression for $[\hat{P}_j, f(\widehat{\vec{X}})]$ involving a partial derivative of f evaluated at $\widehat{\vec{X}}$.

$$\forall \psi \in L^2(\mathbb{R}^N), \quad ([\hat{P}_j, f(\hat{\vec{x}})]\psi)(\vec{x}) = \langle \vec{x} | (\hat{P}_j f(\hat{\vec{x}}) - f(\hat{\vec{x}})\hat{P}_j) | \psi \rangle$$

$$= -i\hbar \frac{\partial}{\partial x_j}(f(\vec{x})\langle \vec{x} | \psi \rangle) - [f(\vec{x})(-i\hbar \frac{\partial}{\partial x_j})\langle \vec{x} | \psi \rangle]$$

$$= -i\hbar \frac{\partial f(\vec{x})}{\partial x_j} \langle \vec{x} | \psi \rangle$$

$$= \langle \vec{x} | (-i\hbar \frac{\partial f(\vec{x})}{\partial x_j}\Big|_{\vec{x} \to \hat{\vec{x}}} | \psi \rangle$$

$$= (-i\hbar \frac{\partial}{\partial x_j} f(\vec{x})\Big|_{\vec{x} \to \hat{\vec{x}}} \psi)(x)$$

$$\Rightarrow \boxed{[\hat{P}_j, f(\hat{\vec{x}})] = -i\hbar \frac{\partial}{\partial \hat{x}_j} f(\hat{\vec{x}})}$$

1.c (5 points) Let $g : \mathbb{R}^N \to \mathbb{R}$ be a differentiable function, and $\widehat{\vec{P}} := (\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_N)$. Derive an expression for $[\hat{X}_j, f(\widehat{\vec{P}})]$ involving a partial derivative of g evaluated at $\widehat{\vec{P}}$.

$$\forall \psi \in L^2(\mathbb{R}^N)$$

$$[\hat{X}_j, f(\hat{\vec{P}})]|\psi\rangle = \int_{\mathbb{R}^N} d^N\vec{p}\, |\vec{p}\rangle\langle\vec{p}| (\hat{X}_j f(\hat{\vec{P}}) - f(\hat{\vec{P}})\hat{X}_j) |\psi\rangle$$

$$= \int_{\mathbb{R}^N} d^N\vec{p}\, |\vec{p}\rangle [i\hbar \frac{\partial}{\partial p_j}(f(\vec{p})\langle \vec{p}|\psi\rangle) - f(\vec{p})\, i\hbar \frac{\partial}{\partial p_j}\langle \vec{p}|\psi\rangle]$$

$$= \int_{\mathbb{R}^N} d^N\vec{p}\, |\vec{p}\rangle [i\hbar \frac{\partial}{\partial p_j} f(\vec{p})]\langle \vec{p}|\psi\rangle$$

$$= \underbrace{\int_{\mathbb{R}^N} d^N\vec{p}\, |\vec{p}\rangle\langle\vec{p}|}_{\hat{I}}\; i\hbar \frac{\partial}{\partial p_j} f(\vec{p})\Big|_{\vec{p} \to \hat{\vec{P}}} |\psi\rangle$$

$$= (i\hbar \frac{\partial}{\partial p_j} f(\vec{p})\Big|_{\vec{p} \to \hat{\vec{P}}})|\psi\rangle$$

$$\Rightarrow \boxed{[\hat{x}_j, f(\hat{\vec{P}})] = i\hbar \frac{\partial}{\partial \hat{P}_j} f(\hat{\vec{P}})}$$

1.d (5 points) Let $\vec{a} \in \mathbb{R}^N$. Derive an expression for $e^{i\vec{a}\cdot\widehat{\vec{P}}/\hbar}\hat{X}_j e^{-i\vec{a}\cdot\widehat{\vec{P}}/\hbar}$ that does not involve $\widehat{\vec{P}}$ or its components.

$$e^{i\vec{a}\cdot\hat{\vec{P}}/\hbar}\hat{X}_j e^{-i\vec{a}\cdot\hat{\vec{P}}/\hbar} = \left([e^{i\vec{a}\cdot\hat{\vec{P}}/\hbar}, \hat{X}_j] + \hat{X}_j e^{i\vec{a}\cdot\hat{\vec{P}}/\hbar}\right) e^{-i\vec{a}\cdot\hat{\vec{P}}/\hbar}$$

$$= -i\hbar\left(\frac{i a_j}{\hbar}\hat{I}\right) + \hat{X}_j$$

$$= \hat{X}_j + a_j\hat{I}$$

1.e (10 points) Let $\hat{u}(t) := e^{i\vec{a}(t)\cdot\widehat{\vec{P}}/\hbar}$ represent a time-dependent translation in $L^2(\mathbb{R}^N)$, where $t \in \mathbb{R}$ and $\vec{a} : \mathbb{R} \to \mathbb{R}^N$ be a continuous function. Consider the quantum system described by the Hilbert space $L^2(\mathbb{R}^N)$ and a standard Hamiltonian, $\hat{H} = \frac{1}{2m}\widehat{\vec{P}}^2 + V(\widehat{\vec{X}})$, where m is a positive real parameter and $V : \mathbb{R}^N \to \mathbb{R}$ is a smooth function. Find an explicit expression for the translated Hamiltonian corresponding to the translation $\psi(t) \to \hat{u}(t)\psi(t)$. To get proper credit, you must simplify the expression for the translated Hamiltonian as much as possible.

$$\tilde{\psi}(t) := \hat{u}(t)\psi(t) \Rightarrow i\hbar\frac{d}{dt}\tilde{\psi}(t) = \tilde{H}(t)\tilde{\psi}(t)$$

$$\Rightarrow i\hbar\left(\frac{d}{dt}\hat{u}\right)\psi + \hat{u}\underbrace{\left(i\hbar\frac{d\psi}{dt}\right)}_{H\psi} = \tilde{H}u\psi$$

$$\Rightarrow \tilde{H} = \hat{u}H\hat{u}^{-1} + \underbrace{\left(i\hbar\frac{d}{dt}\hat{u}\right)}_{i\hbar\left(i\dot{\vec{\alpha}}\cdot\frac{\hat{\vec{P}}}{\hbar}\right)\hat{u}}\hat{u}^{-1}$$

$$\Rightarrow \tilde{H} = \hat{u}H\hat{u}^{-1} - \dot{\vec{\alpha}}\cdot\hat{\vec{P}}$$

$$uH\hat{u}^{-1} = u\left(\frac{\hat{\vec{P}}^2}{2m} + V(\hat{\vec{X}})\right)u^{-1}$$

$$= \frac{\hat{\vec{P}}^2}{2m} + V(\hat{\vec{X}} + \vec{a}\hat{I})$$

$$\Rightarrow \boxed{\tilde{H} = \frac{\hat{\vec{P}}^2}{2m} + V(\hat{\vec{X}} + \vec{a}\hat{I}) - \dot{\vec{\alpha}}(t)\cdot\hat{\vec{P}}}$$

Problem 2 Consider a quantum system corresponding to a particle of mass m that is described by the Hilbert space $L^2(\mathbb{R})$ and the Hamiltonian $\hat{H}_\alpha := \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + \alpha\hat{X}$, where $\hat{X}$ and $\hat{P}$ are standard position and momentum operators acting in $L^2(\mathbb{R})$, ω and α are real numbers, and $\omega > 0$.

2.a (5 points) Find a time-independent translation that maps $\hat{H}_\alpha$ to the simple harmonic oscillator Hamiltonian $\hat{H}_0$.

$$\hat{H}_\alpha = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{X}^2 + \frac{2\alpha}{m\omega^2}\hat{X}\right)$$

$$= \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\left[\left(\hat{X} + \frac{\alpha}{m\omega^2}\hat{I}\right)^2 - \frac{\alpha^2}{m^2\omega^4}\hat{I}\right]$$

Let $\hat{U} = e^{\frac{i a \hat{P}}{\hbar}}$ with $a := \frac{\alpha}{m\omega^2}$, then

$$\hat{U}\hat{X}\hat{U}^{-1} = \hat{X} + a\hat{I}$$

$$\Rightarrow \quad \hat{H}_\alpha = \hat{U}\left[\frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega\hat{X}^2 - \frac{\alpha^2}{2m\omega^2}\hat{I}\right]\hat{U}^{-1}$$

$$= \hat{U}\hat{H}_0\hat{U}^{-1} - \frac{\alpha^2}{2m\omega^2}\hat{I}$$

2.b (5 points) Determine the energy spectrum of $\hat{H}_\alpha$.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{\alpha^2}{2m\omega^2} \qquad n \in \{0, 1, 2, \ldots\} =: \mathbb{N}$$

Becaun $\quad \mathrm{Spec}(\hat{U}\hat{H}_0\hat{U}^{-1}) = \mathrm{Spec}(\hat{H}_0) = \left\{\hbar\omega\left(n+\frac{1}{2}\right) \,\middle|\, n \in \mathbb{N}\right\}$

$\Downarrow$

$$E \in \mathrm{Spec}\, H_\alpha \iff E + \frac{\alpha^2}{2m\omega^2} \in \mathrm{Spec}(\hat{H}_0)$$

$$E = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{\alpha^2}{2m\omega^2}.$$

2.c (5 points) Find values of α such that the ground state energy of the system having $\hat{H}_\alpha$ as its Hamiltonian is zero.

$$E_0 = 0 \Rightarrow \frac{\hbar\omega}{2} - \frac{\alpha^2}{2m\omega^2} = 0$$

$$\Rightarrow \quad \alpha^2 = \hbar m\omega^3 \quad \Rightarrow \quad \boxed{\alpha = \pm\sqrt{\hbar m\omega^3}}$$

Problem 3 Consider a quantum system corresponding to a particle of mass m that is described by the Hilbert space $L^2(\mathbb{R}^2)$ and the Hamiltonian $\hat{H} := \hat{H}_1 + \hat{H}_2$, where for each $j \in \{1, 2\}$, $\hat{H}_j := \frac{1}{2}\hbar\omega_j\{\hat{a}_j, \hat{a}_j^\dagger\}$, $\hat{a}_j := \sqrt{\frac{m\omega_j}{2\hbar}}\hat{X}_j + \frac{i}{\sqrt{2\hbar m\omega_j}}\hat{P}_j$, and ω_j are positive real parameters.

3.a (10 points) Determine the ground state energy of the system. Justify your response.

$[\hat{H}_1, \hat{H}_2] = \hat{0} \Rightarrow \exists$ a complete set of common eigenvectors of $\hat{H}_1$ & $\hat{H}_2$ that form an orthonormal basis of $L^2(\mathbb{R}^n)$.

Because $\hat{H}_1$ & $\hat{H}_2$ are simple harmonic oscillators the elements of this basis has the form

$|n_1, n_2\rangle$, $n_1, n_2 \in \mathbb{N}$ where

$\hat{H}_1 |n_1, n_2\rangle = \hbar\omega_1 (n_1 + \frac{1}{2}) |n_1, n_2\rangle$

$\hat{H}_2 |n_1, n_2\rangle = \hbar\omega_2 (n_2 + \frac{1}{2}) |n_1, n_2\rangle$

$$\Rightarrow \hat{H} |n_1, n_2\rangle = \underbrace{\hbar[\omega_1(n_1 + \tfrac{1}{2}) + \omega_2(n_2 + \tfrac{1}{2})]}_{\text{Energy eigenvalues } = E} |n_1, n_2\rangle$$

Ground state energy: $\boxed{E_0 = \frac{\hbar(\omega_1 + \omega_2)}{2}}$.
$(n_1 = n_2 = 0)$

3.b (10 points) For the case that $\omega_2 = 3\omega_1/4$. Find the degenerate energy eigenvalues of $\hat{H}$ with lowest [redacted] energies. Justify your response.

$\omega_2 = \frac{3\omega_1}{4} \Rightarrow E = \hbar\omega_1 [(n_1 + \frac{1}{2}) + \frac{3}{4}(n_2 + \frac{1}{2})]$

$\Rightarrow E = \frac{\hbar\omega_1}{4}(4n_1 + 3n_2) + \frac{7\hbar\omega_1}{8}$

$\Rightarrow E$ is degenerate if there are $n_1' \neq n_1$ & $n_2' \neq n_2$ such that $4n_1' + 3n_2' = 4n_1 + 3n_2$

$\Rightarrow 4(n_1' - n_1) = 3(n_2 - n_2') \Rightarrow 3$ divides $n_1' - n_1 \Rightarrow$
There is a positive integer k such that $\boxed{n_1' - n_1 = 3k}$

$\Rightarrow \boxed{n_2 - n_2' = 4k} \Rightarrow \boxed{\begin{array}{l} n_1' = n_1 + 3k \\ n_2' = n_2 - 4k \end{array}}$

$$\Rightarrow \begin{cases} n_1' = n_1 + 3,\ n_1 + 6, \ldots \\ n_2' = n_2 - 4,\ n_2 - 8, \ldots \end{cases}$$

lowest energy degenerate level: $(n_1 = 0, n_2 = 4)$, $(n_1' = 3, n_2' = 0)$ $\Rightarrow E = (3 + \frac{7}{8})\hbar\omega_1 = \frac{31}{8}\hbar\omega_1$

Problem 4 Consider a quantum system with Hilbert space $L^2(\mathbb{R}^2)$. Let $\hat{X}_j$, $\hat{P}_j$, $\hat{L}$, and $\mathcal{T}$ respectively denote the standard position, momentum, angular momentum, and time-reversal operators, i.e., $\hat{X}_j$ and $\hat{P}_j$ are as defined in Problem 1, $\hat{L} := \hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1$, and for each $\psi \in L^2(\mathbb{R}^2)$, $(\mathcal{T}\psi)(\vec{x}) := \psi(\vec{x})^*$. $\Rightarrow \quad \langle \vec{x}|\mathcal{T}|\psi\rangle = \langle \vec{x}|\psi\rangle^*$

4.a (5 points) Show that $[\hat{X}_j, \mathcal{T}] = \hat{0}$.

$$\forall \psi \in L^2(\mathbb{R}^2),\ ([\hat{X}_j, \mathcal{T}]\psi)(\vec{x}) = \langle \vec{x}|(\hat{X}_j\mathcal{T} - \mathcal{T}\hat{X}_j)|\psi\rangle$$

$$= x_j \langle \vec{x}|\mathcal{T}|\psi\rangle - \langle \vec{x}|\mathcal{T}\hat{X}_j|\psi\rangle$$

$$= x_j \psi(\vec{x})^* - \langle \vec{x}|\mathcal{T}\hat{X}_j \int_{\mathbb{R}^2} d^2\vec{x}' |\vec{x}'\rangle\langle \vec{x}'|\psi\rangle$$

$$= x_j \psi(\vec{x})^* - \langle \vec{x}|\mathcal{T}\left[\int_{\mathbb{R}^2} d^2\vec{x}'\ x_j' \psi(\vec{x}')|\vec{x}'\rangle\right]$$

$$= x_j \psi(\vec{x})^* - \left[\int_{\mathbb{R}^2} d^2\vec{x}'\ x_j'\psi(\vec{x}')\underbrace{\langle \vec{x}|\vec{x}'\rangle}_{\delta(\vec{x}-\vec{x}')}\right]^* = x_j\psi(\vec{x})^* - x_j\psi(\vec{x})^* = 0$$

4.b (5 points) Show that $\{\hat{P}_j, \mathcal{T}\} = 0$, where $\{A, B\}$ denotes the anticommutator of A and B, i.e., $\{A, B\} := AB + BA$. $\forall \psi \in L^2(\mathbb{R}^2)$,

$$(\{\hat{P}_j, \mathcal{T}\}\psi)(\vec{x}) = \langle \vec{x}|(\hat{P}_j\mathcal{T} + \mathcal{T}\hat{P}_j)|\psi\rangle$$

$$= -i\hbar\frac{\partial}{\partial x_j}\langle \vec{x}|\mathcal{T}|\psi\rangle + \langle \vec{x}|\mathcal{T}\int_{\mathbb{R}^2} d^2\vec{x}' |\vec{x}'\rangle\langle \vec{x}'|\hat{P}_j|\psi\rangle$$

$$= -i\hbar\frac{\partial}{\partial x_j}\psi(\vec{x})^* + \langle \vec{x}|\mathcal{T}\int_{\mathbb{R}^2} d^2\vec{x}'\left[-i\hbar\frac{\partial}{\partial x_j'}\psi(\vec{x}')\right]|\vec{x}'\rangle$$

$$= -i\hbar\frac{\partial}{\partial x_j}\psi(\vec{x})^* + \underbrace{\left[\int_{\mathbb{R}^2} d^2\vec{x}'\left(-i\hbar\frac{\partial}{\partial x_j'}\psi(\vec{x}')\right)\underbrace{\langle \vec{x}|\vec{x}'\rangle}_{\delta(\vec{x}-\vec{x}')}\right]^*}_{i\hbar\frac{\partial}{\partial x_j}\psi(\vec{x})^*} = \hat{0}$$

4.c (5 points) Show that $\{\hat{L}, \mathcal{T}\} = \hat{0}$.

$$\{\hat{L}, \mathcal{T}\} = \{\hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1, \mathcal{T}\}$$

$$= \hat{X}_1\underbrace{\hat{P}_2\mathcal{T}}_{-\mathcal{T}\hat{P}_2} - \hat{X}_2\underbrace{\hat{P}_1\mathcal{T}}_{-\mathcal{T}\hat{P}_1} + \underbrace{\mathcal{T}\hat{X}_1}_{\hat{X}_1\mathcal{T}}\hat{P}_2 - \underbrace{\mathcal{T}\hat{X}_2}_{\hat{X}_2\mathcal{T}}\hat{P}_1$$

$$= -\hat{X}_1\mathcal{T}\hat{P}_2 + \hat{X}_2\mathcal{T}\hat{P}_1 + \hat{X}_1\mathcal{T}\hat{P}_2 - \hat{X}_2\mathcal{T}\hat{P}_1$$

$$= \hat{0}.$$

4.d (5 points) Show that for every $\xi, \zeta \in L^2(\mathbb{R}^2)$, $\langle \xi | \mathcal{T} \zeta \rangle = \langle \mathcal{T} \xi | \zeta \rangle^*$.

$$\langle \xi | T \zeta \rangle = \int_{\mathbb{R}^2} d\vec{x}^2 \; \underbrace{\langle \xi | \vec{x} \rangle}_{\xi(\vec{x})^*} \underbrace{\langle \vec{x} | T \zeta \rangle}_{(T\zeta)(\vec{x}) \,=\, \zeta(\vec{x})^*}$$

$$= \int d\vec{x}^2 \; \xi(\vec{x})^* \, \zeta(\vec{x})^*$$

$$= \langle \zeta | T \xi \rangle$$

$$= \langle T \xi | \zeta \rangle^*$$

4.e (10 points) Consider a state of the system that is represented by a real-valued position wave function $\psi(\vec{x})$, so that $\mathcal{T}\psi = \psi$. Show that the expectation value of the angular momentum vanishes in this state.

$$\{\hat{L}, \mathcal{T}\} = \hat{0} \implies \hat{L}\mathcal{T} = -\mathcal{T}\hat{L}$$

$$\implies \hat{L} = -\mathcal{T}\hat{L}\mathcal{T} \quad \text{because} \quad \mathcal{T}^2 = \hat{I}$$

$$\langle\psi|\hat{L}\psi\rangle = -\langle\psi|\mathcal{T}\hat{L}\underbrace{\mathcal{T}\psi}_{\psi}\rangle = -\underbrace{\langle\psi|\mathcal{T}\underbrace{\hat{L}\psi}_{\xi}\rangle}_{\langle\mathcal{T}\psi|\hat{L}\psi\rangle^* \,=\, \langle\psi|\hat{L}\psi\rangle^*} \quad \text{use } \underline{4.d}$$

$$= -\langle\psi|\hat{L}\psi\rangle^*$$

$$= -\langle\hat{L}\psi|\psi\rangle \quad \leftarrow \text{because } \hat{L}^\dagger = \hat{L}$$

$$= -\langle\psi|\hat{L}\psi\rangle$$

$$\implies 2\langle\psi|\hat{L}\psi\rangle = 0 \implies \frac{\langle\psi|\hat{L}\psi\rangle}{\langle\psi|\psi\rangle} = 0. \quad \boxed{\text{V/Z}}$$

4.f (10 bonus points) Is the expectation value of the angular momentum vanishes in states described by a real-valued momentum wave function $\langle \vec{p}|\psi\rangle$? Why?

$$\langle\psi|\hat{L}\psi\rangle = \int_{\mathbb{R}^2} d^2\vec{p} \int_{\mathbb{R}^2} d^2\vec{p}\,' \; \underbrace{\langle\psi|\vec{p}\rangle}_{=\,\langle\vec{p}|\psi\rangle^* = \langle\vec{p}|\psi\rangle} \langle\vec{p}|\hat{L}|\vec{p}\,'\rangle \underbrace{\langle\vec{p}\,'|\psi\rangle}_{=\,\langle\psi|\vec{p}\,'\rangle}$$

$$= \int_{\mathbb{R}^2} d^2\vec{p} \int_{\mathbb{R}^2} d^2\vec{p}\,' \; \langle\psi|\vec{p}\,'\rangle \langle\vec{p}\,'|\hat{L}|\vec{p}\rangle^* \langle\vec{p}|\psi\rangle$$

$$\langle\vec{p}\,'|\hat{L}|\vec{p}\rangle = \langle\vec{p}\,'|(\hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1)|\vec{p}\rangle$$

$$= p_2\langle\vec{p}\,'|\hat{X}_1|\vec{p}\rangle - p_1\langle\vec{p}\,'|\hat{X}_2|\vec{p}\rangle$$

$$= i\hbar\left(p_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial p_2}\right)\langle\vec{p}\,'|\vec{p}\rangle$$

$$\Downarrow$$

$$\langle\vec{p}\,'|\hat{L}|\vec{p}\rangle^* = -\langle\vec{p}\,'|\hat{L}|\vec{p}\rangle$$

$$\langle\psi|\hat{L}\psi\rangle = \int_{\mathbb{R}^2} d^2\vec{p} \int_{\mathbb{R}^2} d^2\vec{p}\,' \left[-\langle\psi|\vec{p}\,'\rangle\langle\vec{p}\,'|\hat{L}|\vec{p}\rangle\langle\vec{p}|\psi\rangle\right]$$

$$= -\langle\psi|\hat{L}|\psi\rangle$$

$$\Downarrow$$

$$\langle\psi|\hat{L}\psi\rangle = 0 \implies \frac{\langle\psi|\hat{L}\psi\rangle}{\langle\psi|\psi\rangle} = 0 . \quad \blacksquare$$