Phys 402: Final Exam

May 22, 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours and 45 minutes.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.

Problem 1 (10 points) Consider a quantum system consisting of a particle of mass m that is described by the Hilbert space $L^2(\mathbb{R}^3)$ and the Hamiltonian operator: $\hat{H} = \frac{\vec{P}^2}{2m} + \hat{v}(\vec{x})$, where v is a real-valued potential. Let ψ_1 and ψ_2 are solutions of the time-dependent Schrödinger equation for this system, and $\rho := \psi_1^* \psi_2$. Find a vector field $\vec{J} : \mathbb{R}^3 \to \mathbb{R}^3$, that satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0.$$

You need to give the explicit formula for $\vec{J}$ in terms of ψ_1 and ψ_2.

Problem 2 Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the Hilbert spaces of a pair of quantum systems that realize the spin-s_1 and spin-s_2 representations of the algebra $so(3)$, i.e., we can identity $\mathscr{H}_\ell$ with the span of $\{|s_\ell, -s_\ell\rangle, |s_\ell, -s_\ell + 1\rangle, \cdots, |s_\ell, s_\ell\rangle\}$, where $|s_\ell, m_\ell\rangle$ are the normalized common eigenvectors of the spin operators $\hat{S}^2$ and $\hat{S}_3$ in their spin-s_ℓ representation, $\ell \in \{1, 2\}$, and we use the subscripts $1, 2$, and 3 to label the Cartesian coordinates x, y, and z. Suppose that we have a two-particle system where $\mathscr{H}_1$ and $\mathscr{H}_2$ represent the Hilbert spaces of the first and second particle, respectively. Let us denote the spin operators acting in $\mathscr{H}_\ell$ by $\hat{S}_j^{(\ell)}$ and the spin operators for the two-particle system by $\hat{\mathcal{S}}_j := \hat{S}_j^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_j^{(2)}$, where $j \in \{1, 2, 3\}$ and $\hat{I}^{(\ell)}$ is the identity operator acting in $\mathscr{H}_\ell$.

2.a (10 points) Show that $[\hat{\mathcal{S}}_i, \hat{\mathcal{S}}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{\mathcal{S}}_k$ for all $i, j \in \{x, y, z\}$. To get full credit you should show the details of every step of your calculation.

2.b (15 points) Find the eigenvalues of $\hat{\mathcal{S}}_3$ and the corresponding eigenvectors.

2.c (10 points) Find a necessary and sufficient condition on s_1 and s_2 under which at least one of the eigenvalues of $\hat{\mathcal{S}}_3$ is degenerate.

Note: According to the statement of Problem 2a, $\hat{\mathcal{S}}_j$ provide a representation of the algebra $so(3)$. This is a unitary representation because $\hat{\mathcal{S}}_j$ are Hermitian. If $\hat{\mathcal{S}}_3$ has degenerate eigenvalues, this representation is reducible. Therefore your response to this problem is a sufficient condition for the reducibility of this representation.

Problem 3 Consider the two-particle system of Problem 2 with $s_1 = s_2 = 1/2$. Let $|m_1, m_2\rangle := |s_1, m_1\rangle \otimes |s_2, m_2\rangle$. Then $\mathscr{B} := \big\{|\tfrac{1}{2}, \tfrac{1}{2}\rangle, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle, |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\big\}$ is an orthonormal basis of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

3.a (10 points) Construct an operator $\hat{\mathfrak{S}}_+ : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1 \otimes \mathscr{H}_2$ with the following properties:

$$\hat{\mathfrak{S}}_+|\tfrac{1}{2}, \tfrac{1}{2}\rangle = 0, \qquad \hat{\mathfrak{S}}_+|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{2}\,\hbar\,|\tfrac{1}{2}, \tfrac{1}{2}\rangle,$$
$$\hat{\mathfrak{S}}_+|-\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{2}\,\hbar\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \qquad \hat{\mathfrak{S}}_+|-\tfrac{1}{2}, \tfrac{1}{2}\rangle = 0.$$

You are asked to express $\hat{\mathfrak{S}}_+$ as a linear combination of $|m_1, m_2\rangle\langle m_1', m_2'|$, with $m_1, m_2, m_1', m_2' \in \{-\tfrac{1}{2}, \tfrac{1}{2}\}$.

3.b (10 points) Let

$$|e_1\rangle := |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \qquad |e_2\rangle := |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \qquad |e_3\rangle := |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \qquad |e_4\rangle := |-\tfrac{1}{2}, \tfrac{1}{2}\rangle,$$

so that $\mathscr{B} = \{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$. Find the matrix representation of $\hat{\mathfrak{S}}_+$ in the basis $\mathscr{B}$, i.e., compute the matrix $\mathbf{S}_+$ with entries $S_{+ij} := \langle e_i|\hat{\mathfrak{S}}_+|e_j\rangle$.

3.c (15 points) Let $\hat{\mathfrak{S}}_1 := \frac{1}{2}(\hat{\mathfrak{S}}_+ + \hat{\mathfrak{S}}_+^\dagger)$, $\hat{\mathfrak{S}}_2 := \frac{1}{2i}(\hat{\mathfrak{S}}_+ - \hat{\mathfrak{S}}_+^\dagger)$, and $\hat{\mathfrak{S}}_3 := \hat{\mathcal{S}}_3$, where $\hat{\mathcal{S}}_3$ is defined in Problem 2, i.e., $\hat{\mathfrak{S}}_3 = \hat{S}_3^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_3^{(2)}$. Find the matrix representation of $\hat{\mathfrak{S}}_1$, $\hat{\mathfrak{S}}_2$, and $\hat{\mathfrak{S}}_3$ in the basis $\mathscr{B}$.

3.d (20 points) Show that $[\hat{\mathfrak{S}}_i, \hat{\mathfrak{S}}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk}\hat{\mathfrak{S}}_k$, i.e., $\hat{\mathfrak{S}}_j$ define a representation of $so(3)$ in the Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Is this an irreducible representation? Why?

Phys 402 : Sol. to Final Exam Problems

Sol. for Problem 1 :

$$-\frac{\hbar^2}{2m}\nabla^2\psi_1 + V\psi_1 = i\hbar\frac{\partial\psi_1}{\partial t} \quad \Rightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\psi_1^* + V\psi_1^* = -i\hbar\frac{\partial}{\partial t}\psi_1^*$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi_2 + V\psi_2 = i\hbar\frac{\partial}{\partial t}\psi_2$$

$$\Rightarrow \frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}(\psi_1^*\psi_2) = \frac{\partial\psi_1^*}{\partial t}\psi_2 + \psi_1^*\frac{\partial\psi_2}{\partial t}$$

$$= \frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi_1^* + V\psi_1^*\right]\psi_2 + \psi_1^*\left[-\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi_2 + V\psi_2\right)\right]$$

$$= \frac{-i\hbar}{2m}\left[\psi_2\nabla^2\psi_1^* - \psi_1^*\nabla^2\psi_2\right]$$

$$= -\frac{i\hbar}{2m}\left[\vec\nabla\cdot(\psi_2\vec\nabla\psi_1^*) - \vec\nabla\psi_2\cdot\vec\nabla\psi_1^* - \vec\nabla\cdot(\psi_1^*\vec\nabla\psi_2) + \vec\nabla\psi_1^*\cdot\vec\nabla\psi_2\right]$$

$$= -\vec\nabla\cdot\left[\frac{i\hbar}{2m}\left(\psi_2\vec\nabla\psi_1^* - \psi_1^*\vec\nabla\psi_2\right)\right]$$

$$\Rightarrow \boxed{\vec J := \frac{i\hbar}{2m}\left(\psi_2\vec\nabla\psi_1^* - \psi_1^*\vec\nabla\psi_2\right)}$$

Problem #2

2.a)

$$[\hat{\mathbb{S}}_i, \hat{\mathbb{S}}_j] = [\hat{S}_i^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_i^{(2)}, \hat{S}_j^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_j^{(2)}]$$

$$= [\hat{S}_i^{(1)} \otimes \hat{I}^{(2)}, \hat{S}_j^{(1)} \otimes \hat{I}^{(2)}] + [\hat{I}^{(1)} \otimes \hat{S}_i^{(2)}, \hat{S}_j^{(1)} \otimes \hat{I}^{(2)}]$$

$$[\hat{S}_i^{(1)} \otimes \hat{I}^{(2)}, \hat{I}^{(1)} \otimes \hat{S}_j^{(2)}] + [\hat{I}^{(1)} \otimes \hat{S}_i^{(2)}, \hat{I}^{(1)} \otimes \hat{S}_j^{(2)}]$$

$$= \hat{S}_i^{(1)} \hat{S}_j^{(1)} \otimes \hat{I}^{(2)} - \hat{S}_j^{(1)} \hat{S}_i^{(1)} \otimes \hat{I}^{(2)} +$$

$$\hat{S}_j^{(1)} \otimes \hat{S}_i^{(2)} - \hat{S}_j^{(1)} \otimes \hat{S}_i^{(2)} +$$

$$\hat{S}_i^{(1)} \otimes \hat{S}_j^{(2)} - \hat{S}_i^{(1)} \otimes \hat{S}_j^{(2)} +$$

$$\hat{I}^{(1)} \otimes \hat{S}_i^{(2)} \hat{S}_j^{(2)} - \hat{I}^{(1)} \otimes \hat{S}_j^{(2)} \hat{S}_i^{(2)}$$

$$= [\hat{S}_i^{(1)}, \hat{S}_j^{(1)}] \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes [\hat{S}_i^{(2)}, \hat{S}_j^{(2)}]$$

$$= \left(\sum_{k=1}^{3} \epsilon_{ijk} \hat{S}_k^{(1)}\right) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \left(\sum_{k=1}^{3} \epsilon_{ijk} \hat{S}_k^{(2)}\right)$$

$$= \sum_{k=1}^{3} \epsilon_{ijk} \left(\hat{S}_k^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_k^{(2)}\right)$$

$$= \sum_{k=1}^{3} \epsilon_{ijk} \hat{\mathbb{S}}_k .$$

2.b) $\hat{S}_3 = \hat{S}_3^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}_3^{(2)}$

let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be an eigenvector of $\hat{S}_3$

$\Rightarrow \exists\, \alpha_{ij} \in \mathbb{C},\ |\psi\rangle = \sum_{i,j} \alpha_{ij} |s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle$

& $\exists\, \beta \in \mathbb{R},\ \hat{S}_3 |\psi\rangle = \beta |\psi\rangle$

$$\Rightarrow \underbrace{\hat{S}_3 \sum_{i,j} \alpha_{ij} |s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle}_{} = \beta \sum_{i,j} \alpha_{ij} |s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle$$

$$\sum_{i,j} \alpha_{ij} \Big[\big(\hat{S}_3^{(1)} |s_1, m_{1i}\rangle\big) \otimes |s_2, m_{2j}\rangle + |s_1, m_{1i}\rangle \otimes \hat{S}_3^{(2)} |s_2, m_{2j}\rangle \Big]$$

$$\|$$

$$\sum_{i,j} \alpha_{ij} (\hbar m_{1i} + \hbar m_{2j}) |s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle$$

$$\Rightarrow \sum_{i,j} \alpha_{ij} [\hbar (m_{1i} + m_{2j}) - \beta] |s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle = 0$$

Because $\{|s_1, m_{1i}\rangle \otimes |s_2, m_{2j}\rangle\}$ is linearly independent,

$\Rightarrow \alpha_{ij} [\hbar (m_{1i} + m_{2j}) - \beta] = 0$ for all i,j

$|\psi\rangle \neq 0 \Rightarrow \exists\, (i,j)$ such that $\alpha_{ij} \neq 0$

$$\boxed{\beta = \hbar (m_{1i} + m_{2j})}$$

$m_{1i} \in \{-s_1, -s_1+1, \dots, s_1\}$
$m_{2j} \in \{-s_2, -s_2+1, \dots, s_2\}$

$\Rightarrow \beta \in \{-s_1 - s_2, -s_1 - s_2 + 1, \dots, s_1 + s_2\}$

2.(c) $\beta = \hbar(m_1 + m_2)$ where

$m_1 \in \{-s_1, -s_1+1, \ldots, s_1\}$

$m_2 \in \{-s_2, -s_2+1, \ldots, s_2\}$

clearly $\beta = \pm\hbar(s_1+s_2)$ are nondegenerate

$\beta = \pm\hbar(s_1+s_2-1)$ corresponds to

$(m_1 = \pm s_1 \;\&\; m_2 = \pm(s_2-1))$ and

$(m_1 = \pm(s_1-1) \;\&\; m_2 = \pm s_2)$

These two cases are possible if both s_1 & s_2 are nonzero. So for $s_1 \neq 0$ & $s_2 \neq 0$, $\hat{S}_3$ has a degenerate eigenvalue.

If $s_1 = 0$, $m_1 = 0 \Rightarrow \beta = \hbar m_2$ is nondegenerate.

If $s_2 = 0$, $m_2 = 0 \Rightarrow \beta = \hbar m_1$ " " .

So the necessary and sufficient condition is that both s_1 and s_2 must be nonzero.

Problem # 3 :

3. a)

$$\hat{\sigma}_+ = \sqrt{2}\hbar \left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}, -\tfrac{1}{2}| + |\tfrac{1}{2}, \tfrac{1}{2}\rangle\langle \tfrac{1}{2}, -\tfrac{1}{2}| \right)$$

3. b)

$$\hat{\sigma}_+ = \sqrt{2}\hbar \left(|e_2\rangle\langle e_3| + |e_1\rangle\langle e_2| \right)$$

$$\Downarrow$$

$$S_+ = [\langle e_i | \hat{\sigma}_+ | e_j\rangle]$$

$$= \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 & 0 \\ 0 & 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.c

$$\mathcal{S}_+^{\dagger} = \sqrt{2}\,\hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\mathcal{S}_j$ denote the matrix rep. of $\hat{\sigma}_j$. Then

$$\mathcal{S}_1 = \frac{1}{2}\left(\mathcal{S}_+ + \mathcal{S}_+^{\dagger}\right) = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{S}_2 = \frac{1}{2i}\left(\mathcal{S}_+ - \mathcal{S}_+^{\dagger}\right) = \frac{-i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \left[\langle e_i | \hat{\sigma}_3 | e_j \rangle\right]$$

$$\hat{\sigma}_3 |e_1\rangle = \hat{\sigma}_3 |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar |e_1\rangle$$

$$\hat{\sigma}_3 |e_2\rangle = \hat{\sigma}_3 |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0$$

$$\hat{\sigma}_3 |e_3\rangle = \hat{\sigma}_3 |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\hbar |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\hbar |e_3\rangle$$

$$\hat{\sigma}_3 |e_4\rangle = \hat{\sigma}_3 |-\tfrac{1}{2}, \tfrac{1}{2}\rangle = 0$$

$$\Rightarrow \quad \mathcal{S}_3 = \begin{bmatrix} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\hbar & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \hbar \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.d) It is sufficient to show that the matrix rep. of $\hat{S}_j$ in the basis $\mathcal{B}$ satisfy

$$[\mathbb{S}_i, \mathbb{S}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \mathbb{S}_k$$

For $i=1, j=2$ & $i=2, j=3$, & $i=3, j=1$. Other cases follows from properties of ϵ_{ijk} and the antisymmetry of the commutator

$\underline{i=1, j=2:}$

$$[\mathbb{S}_1, \mathbb{S}_2] = \left[\frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}, \frac{-i\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}\right]$$

$$= \frac{-i\hbar^2}{2}\left(\begin{bmatrix}-1&0&1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}1&0&1&0\\0&0&0&0\\-1&0&-1&0\\0&0&0&0\end{bmatrix}\right) = \frac{-i\hbar^2}{2}\begin{bmatrix}-2&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&0&0\end{bmatrix}$$

$$= i\hbar^2\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix} = i\hbar\,\mathbb{S}_3 = i\hbar\sum_{k=1}^{3}\epsilon_{12k}\mathbb{S}_k \quad \checkmark$$

$\underline{i=2, j=3:}$

$$[\mathbb{S}_2, \mathbb{S}_3] = \left[\frac{-i\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}, \hbar\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}\right]$$

$$= \frac{-i\hbar^2}{\sqrt{2}}\left(\begin{bmatrix}0&0&0&0\\-1&0&-1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}\right)$$

$$= \frac{-i\hbar^2}{\sqrt{2}}\begin{bmatrix}0&-1&0&0\\-1&0&-1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix} = i\hbar\left(\frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}\right) = i\hbar\,\mathbb{S}_1$$

$$= i\hbar\sum_{k=1}^{3}\epsilon_{23k}\mathbb{S}_k \quad \checkmark$$

$\underline{i=3, j=1:}$

$$[\mathbb{S}_3, \mathbb{S}_1] = \left[\hbar\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}, \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}\right]$$

$$= \frac{\hbar^2}{\sqrt{2}}\left(\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&-1&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\1&0&-1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right) = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}$$

$$= i\hbar\left(-\frac{i\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}\right) = i\hbar\,\mathbb{S}_2 = i\hbar\sum_{k=1}^{3}\epsilon_{31k}\mathbb{S}_k \quad \checkmark$$

This rep. is clearly reducible as the matrix rep. of the $\hat{\sigma}_j$ are block diagonal with 3×3 and 1×1 blocks:

$$\Phi_1 = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_2 = \frac{-i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_3 = \hbar\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that this is the direct sum of a spin-1 and a spin-0 representation.