# Scattering Theory and $\mathcal{PT}$-Symmetry

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**Abstract** We outline a global approach to scattering theory in one dimension that allows for the description of a large class of scattering systems and their $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetries. In particular, we review various relevant concepts such as Jost solutions, transfer and scattering matrices, reciprocity principle, unidirectional reflection and invisibility, and spectral singularities. We discuss in some detail the mathematical conditions that imply or forbid reciprocal transmission, reciprocal reflection, and the presence of spectral singularities and their time-reversal. We also derive generalized unitarity relations for time-reversal-invariant and $\mathcal{PT}$-symmetric scattering systems, and explore the consequences of breaking them. The results reported here apply to the scattering systems defined by a real or complex local potential as well as those determined by energy-dependent potentials, nonlocal potentials, and general point interactions.

## 1 Basic Setup for Elastic Scattering in One Dimension

The theory of the scattering of waves by obstacles or the interactions modelling them rests on the assumption that the strength of the interaction diminishes at large distances, so that in the vicinity of the source and detectors the wave can be safely approximated by a plane wave. A consistent implementation of this assumption requires the existence of solutions of the relevant wave equation that tend to plane waves at spatial infinities. For a time-harmonic scalar wave, $e^{-i\omega t}\psi(x)$, propagating in one dimension, this requirement takes the form of the following asymptotic boundary conditions:

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$$\psi(x) \to \begin{cases} A_-(k)e^{ikx} + B_-(k)e^{-ikx} & \text{for } x \to -\infty, \\ A_+(k)e^{ikx} + B_+(k)e^{-ikx} & \text{for } x \to +\infty, \end{cases} \quad (1)$$

where $A_\pm(k)$ and $B_\pm(k)$ are complex-valued functions of the wavenumber $k$, which we take to be a positive real variable unless otherwise is clear. The factors $e^{ikx}$ and $e^{-ikx}$ appearing in (1) are related to the solutions, $e^{i(kx-\omega t)}$ and $e^{-i(kx+\omega t)}$, of the wave equation in the absence of the interaction. They represent the right- and left-going waves, respectively.

As a principal example, consider the scattering phenomenon described by the Schrödinger equation,

$$-\psi''(x) + v(x)\psi = k^2\psi(x), \quad (2)$$

where $v(x)$ is a real or complex interaction potential. The existence of the solutions of this equation that satisfy (1) restricts the rate at which $|v(x)|$ decays to zero as $x \to \pm\infty$. We can also consider the more general situations where the potential is energy-dependent. For example consider the Helmholtz equation,

$$\psi''(x) + k^2\hat{\varepsilon}(x,k)\psi(x) = 0, \quad (3)$$

which describes the interaction of polarized electromagnetic waves having an electric field of the form $E_0 e^{-i\omega t}\psi(x)$ pointing along the $y$-axis with an isotropic nonmagnetic media represented by a real or complex relative permittivity profile $\hat{\varepsilon}(x,k)$, [4]. We can express (3) in the form (2) provided that we identify $v(x)$ with the energy-dependent optical potential:

$$v(x,k) = k^2[1 - \hat{\varepsilon}(x,k)]. \quad (4)$$

The scattering setup we have outlined above also applies for the scattering of waves described by nonlocal and nonlinear Schrödinger equations [44, 53, 59, 65],

$$-\psi''(x) + \int_{-\infty}^{\infty} V(x,x')\psi(x')dx' = k^2\psi(x), \quad (5)$$

$$-\psi''(x) + V(x,\psi(x))\psi(x) = k^2\psi(x), \quad (6)$$

if the nonlocal and nonlinear potentials, $V(x,x')$ and $V(x,\psi(x))$ decay sufficiently rapidly as $x \to \pm\infty$ so that (5) and (6) admit solutions satisfying (1). This is clearly the case for confined nonlocal and nonlinear interactions [8, 44], where

$$V(x,x') = v(x)\delta(x-x') + F(x,x')\chi_{[a,b]}(x),$$

$$V(x,\psi(x)) = v(x) + F(x,\psi(x))\chi_{[a,b]}(x),$$

$\delta(x)$ stands for the Dirac delta function, $F$ is a complex-valued function of a pair of real or complex variables, $[a, b]$ is a closed interval of real numbers,

$$\chi_{[a,b]}(x) := \begin{cases} 1 \text{ for } x \in [a, b], \\ 0 \text{ for } x \notin [a, b], \end{cases}$$

and we use the symbol ":=" (respectively "=:") to state that the right-hand (respectively left-hand) side is the definition of the left-hand (respectively right-hand.)

Another class of scattering problems that we can treat using our general framework for scalar-wave scattering in one dimension is that of single- or multi-center point interactions [41]. These correspond to scalar waves $\psi(x)$ that satisfy

$$\begin{aligned} &-\psi''(x) = k^2\psi(x) && \text{for} \quad x \in \mathbb{R} \setminus \{c_1, c_2, \cdots, c_n\}, \\ &\begin{bmatrix} \psi(c_j^+) \\ \psi'(c_j^+) \end{bmatrix} = \mathbf{B}_j \begin{bmatrix} \psi(c_j^-) \\ \psi'(c_j^-) \end{bmatrix} && \text{for} \quad j \in \{1, 2, \cdots, n\}, \end{aligned} \tag{7}$$

where $c_1, c_2, \cdots, c_n$ are distinct real numbers representing the interaction centers, for every function $\phi(x)$ the symbols $\phi(c_j^-)$ and $\phi(c_j^+)$ respectively denote the left and right limit of $\phi(x)$ as $x \to c_j$, i.e., $\phi(c_j^\pm) := \lim_{x\to c_j^\pm} \phi(x)$, and $\mathbf{B}_j$ are possibly $k$-dependent $2\times2$ invertible matrices. The point interactions of this type may be used to model electromagnetic interface conditions [50].

The best known example of a single-center point interaction is the delta-function potential $v(x) = \mathfrak{z}\,\delta(x)$ with a coupling constant $\mathfrak{z}$. It corresponds to the choice: $n = 1$, $c_1 = 0$, and

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ \mathfrak{z} & 1 \end{bmatrix}. \tag{8}$$

In a scattering experiment, the incident wave is emitted by its source which is located at one of the spatial infinities $\pm\infty$, and the scattered wave is received by the detectors which are placed at one or both of these infinities. If the source is located at $-\infty$, the incident wave travels towards the region of the space where the interaction has a sizable strength. A part of it passes through this region and reaches the detector at $+\infty$. The other part gets reflected and travels towards the detector at $-\infty$. As a result, the incident and transmitted waves are right-going while the reflected wave is left-going. This scenario is described by a solution $\psi_l(x)$ of the wave equation that has the following asymptotic behavior.

$$\psi_l(x) \to \begin{cases} \mathcal{N}\left[e^{ikx} + \mathfrak{r}_l(k)\, e^{-ikx}\right] & \text{for } x \to -\infty, \\ \mathcal{N}\,\mathfrak{t}_l(k)\, e^{ikx} & \text{for } x \to +\infty, \end{cases} \tag{9}$$

where $\mathcal{N}$ is the amplitude of the incident wave, and $\mathfrak{r}_l(k)$ and $\mathfrak{t}_l(k)$ are complex-valued functions of $k$ that are respectively called the *left reflection and transmission amplitudes*. Similarly, we have the solution $\psi_r(x)$ of the wave equation that corresponds to the scattering of an incident wave that is emitted from a source located at $x = +\infty$. This satisfies

$$\psi_r(x) \to \begin{cases} \mathcal{N}\,\mathfrak{t}_r(k)\,e^{-ikx} & \text{for } x \to -\infty, \\ \mathcal{N}\left[e^{-ikx} + \mathfrak{r}_r(k)\,e^{ikx}\right] & \text{for } x \to +\infty, \end{cases} \tag{10}$$

where $\mathfrak{r}_r(k)$ and $\mathfrak{t}_r(k)$ are respectively the *right reflection and transmission amplitudes*.

Scattering experiments involve the measurement of the reflection and transmission amplitudes, $\mathfrak{r}_{l/r}(k)$ and $\mathfrak{t}_{l/r}(k)$, or their modulus square, $|\mathfrak{r}_{l/r}(k)|^2$ and $|\mathfrak{t}_{l/r}(k)|^2$, which are respectively called the *left/right reflection and transmission coefficients*.[1] By solving a scattering problem we mean the determination of $\mathfrak{r}_{l/r}(k)$ and $\mathfrak{t}_{l/r}(k)$. We sometimes call these the "*scattering data*".

If $\mathfrak{r}_{l/r}(k_0) = 0$ for some wavenumber $k_0 \in \mathbb{R}^+$, we say that the scatterer[2] is *reflectionless from the left/right* or simply *left/right-reflectionless* at $k = k_0$. Similarly we call it *left/right transparent* at $k = k_0$, if $\mathfrak{t}_{l/r}(k_0) = 1$. A scatterer is invisible from the left or right if it is both reflectionless and transparent from that direction. In this case we call it *left/right-invisible*. *Unidirectional reflectionlessness* (respectively *unidirectional invisibility*) refers to situations where a scatterer is reflectionless (respectively invisible) only from the left or right [25]. The reflectionlessness, transparency, and invisibility of a scatterer are said to be *broadband* if they hold for a finite or infinite interval of positive real values of $k$.

If the wave equation is linear, we can scale $\psi_{l/r}$ and work with $\psi_{+/-} := \psi_{l/r}/\mathcal{N}\mathfrak{t}_{l/r}$. These satisfy:

$$\begin{aligned} &\psi_\pm(x) \to e^{\pm ikx} && \text{for } x \to \pm\infty, \\ &\psi_+(x) \to \frac{1}{\mathfrak{t}_l(k)}\,e^{ikx} + \frac{\mathfrak{r}_l(k)}{\mathfrak{t}_l(k)}\,e^{-ikx} && \text{for } x \to -\infty, \\ &\psi_-(x) \to \frac{\mathfrak{r}_r(k)}{\mathfrak{t}_r(k)}\,e^{ikx} + \frac{1}{\mathfrak{t}_r(k)}\,e^{-ikx} && \text{for } x \to +\infty, \end{aligned} \tag{11}$$

and are called the *Jost solutions*. It turns out that the Schrödinger equation (2) admits Jost solutions, if $\int_{-\infty}^{\infty}\sqrt{1+x^2}\,|v(x)|dx < \infty$. This is equivalent to the *Faddeev condition*:

[1] These are occasionally labelled by $T^{l/r}(k)$ and $R^{l/r}(k)$, [44, 59]. Here we refrain from using this notation, because some references use these symbols for the reflection and transmission amplitudes and not their modulus squared [25].

[2] By a scatterer we mean the interaction causing the propagation of a wave differ from that of a plane wave.

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$$\int_{-\infty}^{\infty}(1+|x|)|v(x)|dx < \infty. \tag{12}$$

Under this condition the Jost solutions exist not only for real and positive values of $k$, but also for complex values of $k$ belonging to the upper-half complex plane, i.e., $k \in \{\mathfrak{z} \in \mathbb{C} \mid \mathrm{Im}(\mathfrak{z}) \geq 0\}$. Furthermore, in this half-plane they are continuous functions of $k$, [21].

Faddeev condition clearly holds for *finite-range potentials* which vanish outside a finite interval (have a compact support), and *exponentially decaying potentials* which satisfy

$$e^{\mu_\pm |x|}|v(x)| < \infty \quad \text{for} \quad x \to \pm\infty, \tag{13}$$

for some $\mu_\pm \in \mathbb{R}^+$. Notice that finite-range potentials fulfill this condition for all $\mu_\pm \in \mathbb{R}^+$. Therefore they share the properties of exponentially decaying potentials that follow from (13).

In this article we use the term "*scattering potential*" for real or complex-valued potentials $v(x)$ that satisfy the Faddeev condition (12).

## 2 Transfer Matrix

Consider a linear scalar wave equation that admits time-harmonic solutions $e^{-i\omega t}\psi(x)$ fulfilling the asymptotic boundary conditions (1). We can identify these solutions by the pairs of column vectors:

$$\begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix}.$$

The $2 \times 2$ matrix $\mathbf{M}(k)$ that connects these is called the *transfer matrix* [63, 66]. By definition, it satisfies

$$\mathbf{M}(k)\begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix} = \begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix}. \tag{14}$$

If we demand that the knowledge of the solution of the wave equation at either of the spatial infinities, $x \to \pm\infty$, determines it uniquely, $\mathbf{M}(k)$ must be invertible. In what follows we assume that this is the case, i.e., $\det \mathbf{M}(k) \neq 0$.[3]

We can express the entries of $\mathbf{M}(k)$ in terms of the reflection and transmission amplitudes by implementing (14) for the Jost solutions. This requires the

[3]In Sect. 4, we prove that this conditions holds for the scattering systems described by the Schrödinger equation (2).

identification of the coefficient functions $A_\pm(k)$ and $B_\pm(k)$ for $\psi(x) = \psi_\pm(x)$. Comparing (1) and (11), we see that for $\psi(x) = \psi_+(x)$,

$$A_- = \frac{1}{\mathfrak{t}_l}, \qquad B_- = \frac{\mathfrak{r}_l}{\mathfrak{t}_l}, \qquad A_+ = 1, \qquad B_+ = 0. \tag{15}$$

Here and in what follows we occasionally suppress the $k$-dependence of $A_\pm(k)$, $B_\pm(k)$, $\mathfrak{r}_{l/r}(k)$, $\mathfrak{t}_{l/r}(k)$, $\mathbf{M}(k)$, and other relevant quantities for brevity. Similarly for $\psi(x) = \psi_-(x)$, we have

$$A_- = 0, \qquad B_- = 1, \qquad A_+ = \frac{\mathfrak{r}_r}{\mathfrak{t}_r}, \qquad B_+ = \frac{1}{\mathfrak{t}_r}. \tag{16}$$

Substituting (15) and (16) in (14) gives

$$\frac{1}{\mathfrak{t}_l}\mathbf{M}\begin{bmatrix} 1 \\ \mathfrak{r}_l \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{M}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\mathfrak{t}_r}\begin{bmatrix} \mathfrak{r}_r \\ 1 \end{bmatrix}. \tag{17}$$

The second of these equations implies

$$M_{12} = \frac{\mathfrak{r}_r}{\mathfrak{t}_r}, \qquad M_{22} = \frac{1}{\mathfrak{t}_r}. \tag{18}$$

Using these relations in the first equation in (17), we find

$$M_{11} = \mathfrak{t}_l - \frac{\mathfrak{r}_l \mathfrak{r}_r}{\mathfrak{t}_r}, \qquad M_{21} = -\frac{\mathfrak{r}_l}{\mathfrak{t}_r}. \tag{19}$$

In view of (18) and (19),

$$\mathbf{M} = \frac{1}{\mathfrak{t}_r}\begin{bmatrix} \mathfrak{t}_l\mathfrak{t}_r - \mathfrak{r}_l\mathfrak{r}_r & \mathfrak{r}_r \\ -\mathfrak{r}_l & 1 \end{bmatrix}. \tag{20}$$

In particular,

$$\det \mathbf{M} = \frac{\mathfrak{t}_l}{\mathfrak{t}_r}. \tag{21}$$

We can also solve (18) and (19) for the reflection and transmission amplitudes in terms of $M_{ij}$. The result is

$$\mathfrak{r}_l = -\frac{M_{21}}{M_{22}}, \qquad \mathfrak{t}_l = \frac{\det \mathbf{M}}{M_{22}}, \qquad \mathfrak{r}_r = \frac{M_{12}}{M_{22}}, \qquad \mathfrak{t}_r = \frac{1}{M_{22}}. \tag{22}$$

Equations (20) and (22) show that *the knowledge of the transfer matrix is equivalent to solving the scattering problem*. It is also instructive to make the $k$-dependence of the Jost solutions explicit and note that in light of (22) and (11) their asymptotic expression takes the form

$$\begin{aligned} &\psi_\pm(k,x) \to e^{\pm ikx} && \text{for } x \to \pm\infty, \\ &\psi_+(k,x) \to \det \mathbf{M}(k)^{-1}\left[M_{22}(k)\, e^{ikx} - M_{21}(k)\, e^{-ikx}\right] && \text{for } x \to -\infty, \\ &\psi_-(k,x) \to M_{12}(k)\, e^{ikx} + M_{22}(k)\, e^{-ikx} && \text{for } x \to +\infty. \end{aligned} \tag{23}$$

These relations together with the assumption that $\det \mathbf{M}(k) \neq 0$ show that *as functions of $k$ the entries of the transfer matrix, $M_{ij}(k)$, have the same analytic properties as the Jost solutions $\psi_\pm(k,x)$.*

A simple consequence of (21) is that $\det \mathbf{M}$ is a measure of the violation of reciprocity in transmission; *a scattering system has reciprocal transmission if and only if* $\det \mathbf{M}(k) = 1$ *for all* $k \in \mathbb{R}^+$.

An example of a scattering system that has nonreciprocal transmission is a single-center point interaction (7) that is defined by a matching matrix $\mathbf{B}_1$ with $\det \mathbf{B}_1 \neq 1$, [41]. To see this, we set $n = 1$ and drop the subscript 1 in $c_1$ and $\mathbf{B}_1$ in (7). Clearly for $x \neq c$, every solution of (7) has the form

$$\psi(x) = A_\pm(k)e^{ikx} + B_\pm(k)e^{-ikx} \quad \text{for} \quad \pm(x-c) > 0. \tag{24}$$

We can use this expression to show that

$$\begin{bmatrix} \psi(c^\pm) \\ \psi'(c^\pm) \end{bmatrix} = \mathbf{N}_c \begin{bmatrix} A_\pm \\ B_\pm \end{bmatrix}, \tag{25}$$

where

$$\mathbf{N}_c(k) := \begin{bmatrix} e^{ick} & e^{-ick} \\ ike^{ick} & -ike^{-ick} \end{bmatrix}. \tag{26}$$

If we substitute (26) in (7), we can relate $A_+(k)$ and $B_+(k)$ to $A_-(k)$ and $B_-(k)$. This gives (14) with the following formula for the transfer matrix of the system.

$$\mathbf{M} = \mathbf{N}_c^{-1}\mathbf{B}\,\mathbf{N}_c. \tag{27}$$

In particular $\det \mathbf{M} = \det \mathbf{B}$. Therefore, single-center point interactions that satisfy $\det \mathbf{B} \neq 1$ violate reciprocity in transmission. These are called *anomalous point interactions* in [41], because they cannot be viewed as singular limits of sequences of scattering potentials.

Next, consider a situation that the solutions $\psi(x)$ of our linear wave equation have also the form of a plane wave in a closed interval, $[x_1, x_1 + \epsilon]$, where $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$, i.e., there are coefficient functions $A_1(k)$ and $B_1(k)$ such that for all $x \in [x_1, x_1 + \epsilon]$,

$$\psi(x) = A_1(k)e^{ikx} + B_1(k)e^{-ikx}. \tag{28}$$

In the limit $\epsilon \to 0$ this is certainly true for any $x_1$, because we can satisfy (28) for $x \to x_1$ by setting

$$A_1(k) = \frac{e^{-ikx}}{2}\left[\psi(x_1) + \frac{\psi'(x_1)}{ik}\right], \quad B_1(k) = \frac{e^{ikx}}{2}\left[\psi(x_1) - \frac{\psi'(x_1)}{ik}\right]. \tag{29}$$

We can use $x_1$ to disect the original scattering problem into two pieces. First, we consider the case where $\psi(x)$ solves the given wave equation for all $x < x_1$ and takes the form (28) for $x \geq x_1$. Then the choice (29) for $A_1(k)$ and $B_1(k)$ ensures the continuity and differentiability of the resulting wave function, namely

$$\psi_1(x) := \begin{cases} \psi(x) & \text{for } x \leq x_1, \\ A_1(k)e^{ikx} + B_1(k)e^{-ikx} & \text{for } x > x_1, \end{cases} \tag{30}$$

at $x = x_1$. We can therefore view $\psi_1(x)$ as the general solution of the wave equation with the interaction terms missing for $x > x_1$. Similarly, we introduce

$$\psi_2(x) := \begin{cases} A_1(k)e^{ikx} + B_1(k)e^{-ikx} & \text{for } x < x_1, \\ \psi(x) & \text{for } x \geq x_1, \end{cases} \tag{31}$$

and identify it with the general solution of the wave equation with the interaction terms missing for $x < x_1$. According to (1), (30), and (31),

$$\psi_1(x) \to \begin{cases} A_-(k)e^{ikx} + B_-(k)e^{-ikx} & \text{for } x \to -\infty, \\ A_1(k)e^{ikx} + B_1(k)e^{-ikx} & \text{for } x \to +\infty, \end{cases} \tag{32}$$

$$\psi_2(x) \to \begin{cases} A_1(k)e^{ikx} + B_1(k)e^{-ikx} & \text{for } x \to -\infty, \\ A_+(k)e^{ikx} + B_+(k)e^{-ikx} & \text{for } x \to +\infty. \end{cases} \tag{33}$$

We can use these relations together with the definition of the transfer matrix to introduce the transfer matrices $\mathbf{M}_j$ for $\psi_j(x)$. These fulfil

$$\mathbf{M}_1\begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}, \qquad \mathbf{M}_2\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix}. \tag{34}$$

Comparing these equations with (14), we see that the transfer matrix of the original wave equation is given by

$$\mathbf{M} = \mathbf{M}_2\mathbf{M}_1. \tag{35}$$

Now, consider dividing the set of real numbers into $n + 1$ intervals:

$$I_1 := (-\infty, a_1], \quad I_2 := [a_1, a_2], \quad I_3 := [a_2, a_3], \quad \cdots, \quad I_n := [a_{n-1}, a_n],$$
$$I_{n+1} := [a_n, \infty),$$

and let $\mathbf{M}_j$ be the transfer matrix for the scattering of a scalar wave with interactions confined to $I_j$. Then a repeated use of the argument leading to (35) shows that the transfer matrix for the original scattering problem is given by

$$\mathbf{M} = \mathbf{M}_{n+1}\mathbf{M}_n\mathbf{M}_{n-1}\cdots\mathbf{M}_1. \tag{36}$$

This property, which is known as the *composition rule for the transfer matrices*, allows for reducing the scattering problem with interactions taking place in an arbitrary region of space to simpler scattering problems where the interaction is confined to certain intervals.

For example, if the interaction has a finite range, i.e., it seizes to exist outside an interval $[a, b]$, we can set

$$a_j := a + \frac{(j-1)(b-a)}{n} \quad \text{for} \quad j = 1, 2, \cdots, n.$$

In this way, by taking large values for $n$ we can reduce the initial scattering problem to those whose solution requires solving the wave equation in small intervals. If the interaction is a smooth function of space, we can approximate it by a constant in each of these intervals. This in turn simplifies the calculation of $\mathbf{M}_j$. We can use the result of this calculation together with (36) to find an approximate expression for $\mathbf{M}$. Aside from the technical problems of multiplying a large number of $2 \times 2$ matrices, this provides a simple approach for the solution of the scattering problem for finite-range linear interactions.

We can easily implement this procedure to solve the scattering problem for a multi-center point interaction (7). To do this we label the centers of the point interaction so that $c_1 < c_2 < \cdots < c_n$ and compute the transfer matrix for single-center point interactions associated with $c_j$. As we explained above this has the form

$$\mathbf{M}_j = \mathbf{N}_j^{-1}\mathbf{B}_j\mathbf{N}_j, \tag{37}$$

where $\mathbf{N}_j$ is given by the right-hand side of (26) with $c$ changed to $c_j$. We can then determine the transfer matrix of the multi-center point interaction by invoking the composition rule (36). The result is

$$\mathbf{M} = \mathbf{N}_n^{-1}\mathbf{B}_n\mathbf{N}_n\mathbf{N}_{n-1}^{-1}\mathbf{B}_{n-1}\mathbf{N}_{n-1}\cdots\mathbf{N}_1^{-1}\mathbf{B}_1\mathbf{N}_1. \tag{38}$$

In particular, we find that

$$\det\mathbf{M} = \det\mathbf{B}_1\det\mathbf{B}_2\cdots\det\mathbf{B}_n. \tag{39}$$

Combing this equation with (21), we infer that *a multi-center point interaction violates reciprocity in transmission if and only if it consists of an odd number of anomalous single-center point interactions*.

Next, consider a multi-delta-function potential

$$v(x)=\epsilon\sum_{j=1}^{n}\mathfrak{z}_j\delta(x-c_j), \tag{40}$$

where $\epsilon$ is a nonzero real parameter and $\mathfrak{z}_j$ are possibly complex coupling constants. We can identify this with the multi-center point interaction with matching matrices

$$\mathbf{B}_j=\begin{bmatrix}1 & 0\\ \epsilon\,\mathfrak{z}_j & 1\end{bmatrix}. \tag{41}$$

Substituting this relation in (38) we find the transfer matrix $\mathbf{M}$ for (40). This has a unit determinant, because $\det\mathbf{B}_j=1$ and $\mathbf{M}$ satisfies (39).

It is not difficult to see that the transfer matrix $\mathbf{M}$ of the multi-delta-function potential (40) and hence its entries are polynomials of degree at most $n$ in the parameter $\epsilon$. In view of (23), and the fact that $\det\mathbf{M}=1$, this implies that the same is true of the Jost solutions of the Schrödinger equation (2) for this potential. This observation shows that if we treat $\epsilon$ as a perturbation parameter and perform an $n$-th order perturbative calculation of the Jost solutions, we obtain their exact expression. In view of (11), this allows for determining the reflection and transmission amplitudes of (40). We therefore have the following result.

**Theorem 1** *The n-th order perturbation theory gives the exact solution of the scattering problem for multi-delta-function potentials with n centers.*

In fact, a direct analysis shows that $n$-th order perturbation theory gives the exact solution of the Schrödinger equation (2) for multi-delta-function potentials (40), [54].

## 3 Scattering Matrix

By definition, the scattering operator, which is also known as the scattering matrix, maps the waves traveling toward the interaction region (incoming waves) to those traveling away from it (outgoing waves). In one dimension, the boundary conditions (1) at spatial infinities show that the incoming waves have the asymptotic form $A_-(k)e^{ikx}$ (respectively $B_-(k)e^{-ikx}$), if their source is located at $x=-\infty$ (respectively $x=+\infty$), and the outgoing waves tend to $B_+e^{-ikx}$ as $x\to-\infty$ and $A_+(k)e^{ikx}$ as $x\to+\infty$. In light of these observations, we can quantify the scattering operator by a $2\times 2$ matrix $\mathbf{S}(k)$ that connects $A_-(k)$ and $B_+(k)$ to $A_+(k)$ and $B_-(k)$. Clearly there are four different ways of doing so, namely

$$\mathbf{S}_1\begin{bmatrix}A_-\\B_+\end{bmatrix}=\begin{bmatrix}A_+\\B_-\end{bmatrix},\qquad \mathbf{S}_2\begin{bmatrix}A_-\\B_+\end{bmatrix}=\begin{bmatrix}B_-\\A_+\end{bmatrix},$$
$$\mathbf{S}_3\begin{bmatrix}B_+\\A_-\end{bmatrix}=\begin{bmatrix}A_+\\B_-\end{bmatrix},\qquad \mathbf{S}_4\begin{bmatrix}B_+\\A_-\end{bmatrix}=\begin{bmatrix}B_-\\A_+\end{bmatrix}. \tag{42}$$

These correspond to various conventions for defining the **S**-matrix in one dimension. It is easy to see that

$$\mathbf{S}_2=\boldsymbol{\sigma}_1\mathbf{S}_1,\qquad \mathbf{S}_3=\mathbf{S}_1\boldsymbol{\sigma}_1,\qquad \mathbf{S}_4=\boldsymbol{\sigma}_1\mathbf{S}_1\boldsymbol{\sigma}_1, \tag{43}$$

where $\boldsymbol{\sigma}_1$ is the first Pauli matrix,

$$\boldsymbol{\sigma}_1:=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Next, let us express the entries of $\mathbf{S}_1$ in terms of the reflection and transmission amplitudes. To do this, we implement the first equation in (42) for the Jost solutions $\psi_\pm(x)$. For $\psi(x)=\psi_+(x)$, $A_\pm$ and $B_\pm$ are given by (15). Substituting these in the first equation in (42) gives

$$\mathbf{S}_1\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\mathfrak{t}_l\\\mathfrak{r}_l\end{bmatrix}. \tag{44}$$

Similarly for $\psi(x)=\psi_-(x)$, we use (15) to obtain

$$\mathbf{S}_1\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\mathfrak{r}_r\\\mathfrak{t}_r\end{bmatrix}. \tag{45}$$

In view of Eqs. (44) and (45),

$$\mathbf{S}_1=\begin{bmatrix}\mathfrak{t}_l&\mathfrak{r}_r\\\mathfrak{r}_l&\mathfrak{t}_r\end{bmatrix}. \tag{46}$$

This relation together with (43) imply

$$\mathbf{S}_2=\begin{bmatrix}\mathfrak{r}_l&\mathfrak{t}_r\\\mathfrak{t}_l&\mathfrak{r}_r\end{bmatrix},\qquad \mathbf{S}_3=\begin{bmatrix}\mathfrak{r}_r&\mathfrak{t}_l\\\mathfrak{t}_r&\mathfrak{r}_l\end{bmatrix},\qquad \mathbf{S}_4=\begin{bmatrix}\mathfrak{t}_r&\mathfrak{r}_l\\\mathfrak{r}_r&\mathfrak{t}_l\end{bmatrix}. \tag{47}$$

According to Eqs. (46) and (47), we can use any of $\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3$, and $\mathbf{S}_4$ to encode the information about the scattering properties of the system. They are therefore physically equivalent. We adopt the convention of identifying the **S**-matrix with $\mathbf{S}_1$, i.e., set

$$\mathbf{S} := \begin{bmatrix} \mathfrak{t}_l & \mathfrak{r}_r \\ \mathfrak{r}_l & \mathfrak{t}_r \end{bmatrix}. \tag{48}$$

This choice has the appealing property of reducing to the $2 \times 2$ identity matrix $\mathbf{I}$ in the absence of interactions.

Eigenvalues of the scattering matrix turn out to contain some useful information about the scattering properties of the system. In view of (48), they have the form:

$$\mathfrak{s}_\pm = \frac{\mathfrak{t}_l + \mathfrak{t}_r}{2} \pm \sqrt{\left(\frac{\mathfrak{t}_l - \mathfrak{t}_r}{2}\right)^2 + \mathfrak{r}_l \mathfrak{r}_r}. \tag{49}$$

In particular, whenever $\mathfrak{t}_l = \mathfrak{t}_r =: \mathfrak{t}$,

$$\mathfrak{s}_\pm = \mathfrak{t} \pm \sqrt{\mathfrak{r}_l \mathfrak{r}_r}. \tag{50}$$

Both the transfer and the $\mathbf{S}$-matrix contain complete information about the scattering data, but in contrast to the transfer matrix the $\mathbf{S}$-matrix does not obey a useful composition rule. An advantage of the $\mathbf{S}$-matrix is the simplicity of its higher-dimensional, relativistic, and field theoretical generalizations [76].[4]

## 4 Potential Scattering, Reciprocity Theorem, and Invisibility

Consider the time-independent Schrödinger equation (2) for a scattering potential $v(x)$ which admits Jost solutions $\psi_\pm$ and defines a valid scattering problem. Being solutions of a second order linear homogeneous differential equation, $\psi_\pm$ are linearly independent if and only if their Wronskian, $W(x) := \psi_-(x)\psi_+'(x) - \psi_+(x)\psi_-'(x)$, does not vanish at some $x \in \mathbb{R}$, [5]. In fact, because the Schrödinger equation (2) does not involve the first derivative of $\psi$, $W(x)$ is a constant.[5] We can determine this constant using the asymptotic expression (11) for the $\psi_\pm(x)$. Doing this for $x \to -\infty$ and $x \to +\infty$, we respectively find $W(x) = 2ik/\mathfrak{t}_l(k)$ and $W(x) = 2ik/\mathfrak{t}_r(k)$. This proves the following reciprocity theorem.

**Theorem 2 (Reciprocity in Transmission)** *The left and right transmission amplitudes of every real or complex scattering potential coincide, i.e.,*

$$\mathfrak{t}_l(k) = \mathfrak{t}_r(k). \tag{51}$$

[4] A genuine multidimensional generalization of the transfer matrix has been recently proposed in [31].

[5] This can be easily checked by differentiating $W(x)$ and using (2) to show that $W'(x) = 0$.

In the following we use $\mathfrak{t}(k)$ for the common value of $\mathfrak{t}_l(k)$ and $\mathfrak{t}_r(k)$ whenever a scattering system has reciprocal transmission.

In view of Eqs. (20), (21), (22), (48), and (51), the transfer and scattering matrices and the scattering data associated with real or complex scattering potentials satisfy:

$$\mathbf{M} = \frac{1}{\mathfrak{t}}\begin{bmatrix} \mathfrak{t}^2 - \mathfrak{r}_l\mathfrak{r}_r & \mathfrak{r}_r \\ -\mathfrak{r}_l & 1 \end{bmatrix}, \qquad \det\mathbf{M} = 1, \qquad \mathbf{S} = \begin{bmatrix} \mathfrak{t} & \mathfrak{r}_r \\ \mathfrak{r}_l & \mathfrak{t} \end{bmatrix}, \tag{52}$$

$$\mathfrak{r}_l = -\frac{M_{21}}{M_{22}}, \qquad \mathfrak{r}_r = \frac{M_{12}}{M_{22}}, \qquad \mathfrak{t} = \frac{1}{M_{22}}. \tag{53}$$

Another consequence of (51) is that the Wronskian of the Jost solutions take the form

$$W(x) = \frac{2ik}{\mathfrak{t}(k)}. \tag{54}$$

This is a number depending on the value of $k$. In particular, for $k \in \mathbb{R}^+$ it cannot diverge. This proves the following theorem.

**Theorem 3** *Let $v(x)$ be a real or complex scattering potential. Then its transmission amplitude does not vanish for any wavenumber, i.e.,*

$$\mathfrak{t}(k) \neq 0 \quad \text{for} \quad k \in \mathbb{R}^+. \tag{55}$$

This theorem shows that real and complex scattering potentials can never serve as a perfect absorber. According to Theorem 2 they cannot even serve as an approximate one-way filter.

Next, we examine the following simple example:

$$v(x) = \mathfrak{z}\chi_{[0,L]}(x) = \begin{cases} \mathfrak{z} \text{ for } x \in [0, L], \\ 0 \text{ for } x \notin [0, L], \end{cases} \tag{56}$$

where $\mathfrak{z}$ and $L$ are nonzero complex and real parameters. This is a piecewise constant finite-range potential with support $[0, L]$, which we can identify with a rectangular barrier potential of a possibly complex height $\mathfrak{z}$.

We can easily solve the Schrödinger equation (2) for the barrier potential (56). Its general solution has the form

$$\psi(x) = \begin{cases} A_-(k)e^{ikx} + B_-(k)e^{-ikx} & \text{for} \quad x < 0, \\ A_0(k)e^{ik\mathfrak{n}x} + B_0(k)e^{-ik\mathfrak{n}x} & \text{for } x \in [0, L], \\ A_+(k)e^{ikx} + B_+(k)e^{-ikx} & \text{for} \quad x \geq L, \end{cases} \tag{57}$$

where $A_j(k)$ and $B_j(k)$, with $j = 0, \pm$, are complex-valued coefficient functions,

$$\mathfrak{n} := \sqrt{1 - \frac{\mathfrak{z}}{k^2}}, \tag{58}$$

and for every complex number $w$ we use $\sqrt{w}$ to label the principal value of $w^{1/2}$, i.e., $\sqrt{w} = \sqrt{|w|}e^{i\varphi}$ with $\varphi \in [0, \pi)$. By demanding $\psi$ to be continuous and differentiable at $x = L$ and $x = 0$, we can respectively express $A_+$ and $B_+$ in terms of $A_0$ and $B_0$, and $A_0$ and $B_0$ in terms of $A_-$ and $B_-$. This in turn allows us to relate $A_+$ and $B_+$ to $A_-$ and $B_-$. We can write the resulting equations in the form (14) with the transfer matrix given by

$$\mathbf{M}(k) = \begin{bmatrix} [\cos(kL\mathfrak{n}) + i\mathfrak{n}_+ \sin(kL\mathfrak{n})]e^{-ikL} & i\mathfrak{n}_- \sin(kL\mathfrak{n})e^{-ikL} \\ -i\mathfrak{n}_- \sin(kL\mathfrak{n})e^{ikL} & [\cos(kL\mathfrak{n}) - i\mathfrak{n}_+ \sin(kL\mathfrak{n})]e^{ikL} \end{bmatrix}, \tag{59}$$

and $\mathfrak{n}_\pm := (\mathfrak{n} \pm \mathfrak{n}^{-1})/2$.

In view of (53), we can use (59) to read off the expression for the reflection and transmission amplitudes of the barrier potential (56). These have the form:

$$\mathfrak{r}_l(k) = \frac{i\mathfrak{n}_- \tan(kL\mathfrak{n})}{1 - i\mathfrak{n}_+ \tan(kL\mathfrak{n})}, \tag{60}$$

$$\mathfrak{r}_r(k) = \frac{i\mathfrak{n}_- \tan(kL\mathfrak{n})e^{-2ikL}}{1 - i\mathfrak{n}_+ \tan(kL\mathfrak{n})}, \tag{61}$$

$$\mathfrak{t}(k) = \frac{e^{-ikL}}{\cos(kL\mathfrak{n}) - i\mathfrak{n}_+ \sin(kL\mathfrak{n})}. \tag{62}$$

Clearly, $\mathfrak{t}(k) \neq 0$ for all $k \in \mathbb{R}^+$. We can check that indeed $\det \mathbf{M}(k) = 1$, and evaluate the **S**-matrix and its eigenvalues. In light of (50) the latter are given by

$$\mathfrak{s}_\pm(k) = \left[\frac{1 \pm i\mathfrak{n}_- \tan(kL\mathfrak{n})}{1 - i\mathfrak{n}_+ \tan(kL\mathfrak{n})}\right] e^{-ikL}. \tag{63}$$

According to (60) the barrier potential (56) is left-reflectionless if and only if $\mathfrak{n}$ is real and $k = k_m := \pi m/L\mathfrak{n}$ for a positive integer $m$.[6] In this case it is also right-reflectionless, but not in general transparent. It is easy to show that for these values of the wavenumber, $\mathfrak{t}(k) = e^{-im\pi(\mathfrak{n}^{-1}+1)}$. This equals unity, i.e., the potential is transparent and hence bidirectionally invisible if and only if there is an integer $q$ such that $\mathfrak{n} = (2q/m - 1)^{-1}$. It is easy to see that this is equivalent to demanding that

[6]Equation (58) implies that $k_m = \sqrt{(\pi m/L)^2 + \mathfrak{z}}$. This in turn means that for $\mathfrak{z} > 0$, $m$ can be any positive integer, and for $\mathfrak{z} < 0$, $m > L\sqrt{-\mathfrak{z}}/\pi$.

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$$\mathfrak{z} = \frac{4\pi^2 q(q-m)}{L^2}, \qquad\qquad k = \frac{2q-m}{L}.$$

Because $k > 0$, the latter relation implies that $2q > m$.

The entries of the transfer matrix for the barrier potential (56) are smooth functions of the wavenumber $k$. In fact, we can analytically continue them to the entire complex $k$-plane. This turns out to be a common feature of all finite-range potentials. To see this first we note that if a potential $v(x)$ decays exponentially as $x \to \pm\infty$, i.e., there are positive numbers $\mu_\pm$ satisfying (13), then the Jost solutions are holomorphic (complex analytic) functions in the strip [3]:

$$\mathscr{S}_{\mu_\pm} := \{k \in \mathbb{C} \mid -\mu_- < \mathrm{Im}(k) < \mu_+\}. \tag{64}$$

In light of (23) and the fact that $\det \mathbf{M} = 1$, this implies that the same holds for the entries of the transfer matrix. We state this result as a theorem:

**Theorem 4** *Let $v(x)$ be a real or complex potential satisfying (13) for some $\mu_\pm > 0$. Then the entries $M_{ij}(k)$ of its transfer matrix are holomorphic functions in the strip (64).*

A basic result of complex analysis is that a nonzero holomorphic function can only vanish at a discrete set of isolated points. In view of Theorem 4 this applies to the entries of the transfer matrix of exponentially decaying potentials. In particular, for each choice of $i$ and $j$ in $\{1, 2\}$, either $M_{ij}(k) = 0$ for all $k \in \mathscr{S}_{\mu_\pm}$ or there is a (possibly empty) discrete set of isolated values of $k \in \mathscr{S}_{\mu_\pm}$ at which $M_{ij}(k)$ vanishes. This is particularly important, because $\mathscr{S}_{\mu_\pm}$ contains the positive real axis where the physical wavenumbers reside.

According to (53), the zeros of $M_{12}(k)$ (respectively $M_{21}(k)$) that are located on the positive real axis are the wavenumbers $k_0$ at which the right (respectively left) reflection amplitude of the potential $v(x)$ vanishes, i.e., $v(x)$ is right- (respectively left-) reflectionless at $k_0$. Similarly, if $M_{22}(k_0) = 1$, then $\mathfrak{t}(k_0) = 1$, and $v(x)$ is transparent at $k_0$. Therefore real and positive zeros of $M_{12}(k)$, $M_{21}(k)$, and $M_{22}(k) - 1$ are the wavenumbers at which $v(x)$ is right-reflectionless, left-reflectionless, and transparent. In particular, equations

$$M_{12}(k) = M_{22}(k) - 1 = 0, \tag{65}$$

$$M_{21}(k) = M_{22}(k) - 1 = 0, \tag{66}$$

respectively characterize the invisibility of the potential from the right and left. These results are clearly valid for any scattering system whose scattering features can be described using a transfer matrix.

The following no-go theorem is a simple consequence of Eqs. (53) and the above-mentioned property of the zeros of holomorphic functions.

**Theorem 5** *If the entries $M_{ij}(k)$ of the transfer matrix for a scattering system are nonzero functions that are holomorphic on the positive real axis in the complex $k$-*

*plane, then the system cannot display broadband reflectionlessness, transparency, or invisibility from either direction.*

According to Theorem 4, the conclusion of this theorem applies to exponentially decaying and finite-range potentials.

The above analysis does not exclude the existence of exponentially decaying potentials that are unidirectionally or bidirectionally reflectionless for all $k \in \mathbb{R}^+$ (fullband reflectionlessness). Such potentials were known to exist since the 1930s. The principal example is the Pöschl-Teller potential:

$$v(x) = -\frac{\zeta}{\cosh(\alpha x)},$$

where $\zeta$ and $\alpha$ are positive real parameters. It turns out that the scattering problem for this potential admits an exact solution, and that for integer values of $\zeta/\alpha^2$ it is bidirectionally reflectionless for all $k \in \mathbb{R}^+$, [10]. The Pöschl-Teller potential is a member of an infinite class of real, attractive (negative), exponentially decaying potentials with this property. These were initially obtained in the 1950s as an application of the methods of inverse scattering theory [20]. Their much less-known complex analogs were constructed in the 1990s, [74].[7]

The construction of scattering potentials that are unidirectionally invisible in the entire spectral band is a much more recent development [16, 30]. Before making specific comments about these potentials, we wish to address the problem of the existence of exponentially decaying and finite-range potentials that are unidirectionally reflectionless, transparent, or invisible in the whole spectral band. To do this, first we examine the structure of the transfer matrix $\mathbf{M}(k)$ for negative values of $k$.

Consider a solution of the Schrödinger equation (2) for a scattering potential $v(x)$. In order to make the $k$-dependence of this solution explicit, we denote it by $\psi(k, x)$. In particular, we write (1) as

$$\psi(k, x) \to A_\pm(k)e^{ikx} + B_\pm(k)e^{-ikx} \qquad \text{for} \qquad x \to \pm\infty. \tag{67}$$

Because the Schrödinger equation (2) is invariant under $k \to -k$,

$$\breve{\psi}(k, x) := \psi(-k, x) \tag{68}$$

is also a solution of (2). In view of the fact that $v(x)$ is a scattering potential, $\breve{\psi}(k, x)$ must satisfy the asymptotic boundary conditions:

$$\breve{\psi}(k, x) \to \breve{A}_\pm(k)e^{-ikx} + \breve{B}_\pm(k)e^{ikx} \qquad \text{for} \qquad x \to \pm\infty, \tag{69}$$

[7] Reflectionless potentials also arise as soliton solutions of nonlinear differential equations [24].

where $\breve{A}_\pm(k)$ and $\breve{B}_\pm(k)$ are some coefficient functions. We can use (67), (68), and (69) to show that for $k \in \mathbb{R}^-$,

$$\breve{A}_\pm(k) = B_\pm(-k), \qquad \breve{B}_\pm(k) = A_\pm(-k). \tag{70}$$

Now, suppose that we can analytically continue $\mathbf{M}(k)$ from $k \in \mathbb{R}^+$ to $k \in \mathbb{R}^-$. Then we can relate $\breve{A}_+(k)$ and $\breve{B}_+(k)$ to $\breve{A}_-(k)$ and $\breve{B}_-(k)$ using $\mathbf{M}(k)$ for $k \in \mathbb{R}^-$. This gives

$$\begin{bmatrix} \breve{A}_+(k) \\ \breve{B}_+(k) \end{bmatrix} = \mathbf{M}(k) \begin{bmatrix} \breve{A}_-(k) \\ \breve{B}_-(k) \end{bmatrix}. \tag{71}$$

Substituting (70) in this equation and using (14), we arrive at

$$\mathbf{M}(k) = \boldsymbol{\sigma}_1 \mathbf{M}(-k) \boldsymbol{\sigma}_1, \tag{72}$$

where $k \in \mathbb{R}^-$. Because this equation is invariant under $k \to -k$, it holds for all $k \in \mathbb{R} \setminus \{0\}$. In terms of the components of $\mathbf{M}(k)$, we can write (72) in the form:

$$M_{11}(-k) = M_{22}(k), \qquad M_{12}(-k) = M_{21}(k), \tag{73}$$

which again hold for all $k \in \mathbb{R} \setminus \{0\}$.

Equations (72) and (73) apply to any scattering system in which the wave equation involves even powers of $k$ and have a transfer matrix that can be analytically continued from the positive to the negative real axis in the complex $k$-plane. For such systems, we can determine the reflection and transmission amplitudes for $k \in \mathbb{R}^-$, by inserting (73) in (22). This gives

$$\mathfrak{r}_l(-k) = -\frac{\mathfrak{r}_r(k)}{\mathfrak{D}(k)}, \quad \mathfrak{t}_l(-k) = \frac{\mathfrak{t}_l(k)}{\mathfrak{D}(k)}, \quad \mathfrak{r}_r(-k) = -\frac{\mathfrak{r}_l(k)}{\mathfrak{D}(k)}, \quad \mathfrak{t}_r(-k) = \frac{\mathfrak{t}_r(k)}{\mathfrak{D}(k)}, \tag{74}$$

where

$$\mathfrak{D}(k) := \frac{M_{11}(k)}{M_{22}(k)} = \mathfrak{t}_l(k)\mathfrak{t}_r(k) - \mathfrak{r}_l(k)\mathfrak{r}_r(k) = \det \mathbf{S}(k). \tag{75}$$

Again, because Eqs. (74) are invariant under $k \to -k$, they hold for all $k \in \mathbb{R} \setminus \{0\}$. A straightforward consequence of these equations is that if $\mathfrak{r}_{l/r}(k)$ (respectively $\mathfrak{t}_{l/r}(k)$) vanishes for all $k \in \mathbb{R}^+$, then it will also vanish for all $k \in \mathbb{R}^-$. It is important to note that this conclusion relies on the existence of the analytic continuation of $\mathbf{M}_{ij}(k)$ from $k \in \mathbb{R}^+$ to $k \in \mathbb{R}^-$. Certainly, this condition holds for finite-range and exponentially decaying potentials. This together with Theorem 5 prove the following result.

**Theorem 6** *Scattering potentials with a finite range or an asymptotic exponential decay cannot display broadband unidirectional reflectionlessness, transparency, or invisibility.*

This theorem shows that as far as finite-range and exponentially decaying potentials are concerned, unidirectional reflectionlessness, transparency, and invisibility can only be achieved at a discrete set of isolated values of the wavenumber.

The principal example of a unidirectionally invisible finite-range potential is

$$v(x) = \begin{cases} \mathfrak{z}\, e^{iKx} & \text{for } x \in [-\frac{L}{2}, \frac{L}{2}], \\ 0 & \text{for } x \notin [-\frac{L}{2}, \frac{L}{2}], \end{cases} \tag{76}$$

where $\mathfrak{z}$, $K$, and $L$ are nonzero real parameters, and $L > 0$, [13, 23, 25, 61]. This potential is unidirectionally invisible from the left for the wavenumber $k = K/2$, if $K = 2\pi/L$ and $K^2\mathfrak{z} \ll 1$. It belongs to the class of locally periodic finite-range potentials of the form

$$v(x) = \begin{cases} f(x) & \text{for } x \in [-\frac{L}{2}, \frac{L}{2}], \\ 0 & \text{for } x \notin [-\frac{L}{2}, \frac{L}{2}], \end{cases} \tag{77}$$

where

$$f(x) := \sum_{n=-\infty}^{\infty} \mathfrak{z}_n e^{iK_n x}, \tag{78}$$

$\mathfrak{z}_n$ are complex coefficients, and $K_n := 2\pi n/L$. The following theorem, which is proven in Ref. [46], reveals a remarkable property of these potentials.

**Theorem 7** *Let $v(x)$ be a potential of the form (77) and suppose that we are interested in the scattering of waves of wavenumber $k$ satisfying $|\mathfrak{z}_n|/k^2 \ll 1$, so that the first Born approximation is valid. If $\mathfrak{z}_n = 0$ for all $n \le 0$, $v(x)$ is unidirectionally left-invisible for all $k = K_n/2 = \pi n/L$.*[8]

Now, consider taking $L \to \infty$. Then (77) becomes $v(x) = f(x)$, the Fourier series in (78) turns into a Fourier integral, the role of $\mathfrak{z}_n$ is played by the Fourier transform of $v(x)$, i.e., $\tilde{v}(\mathfrak{K}) := \int_{-\infty}^{\infty} e^{-i\mathfrak{K}x} v(x)dx$, and Theorem 7 states that *if the first Born approximation is reliable, then $v(x)$ is unidirectionally left-invisible for all $k \in \mathbb{R}^+$ provided that $\tilde{v}(\mathfrak{K}) = 0$ for $\mathfrak{K} \le 0$.* A highly nontrivial observation is that the same conclusion may be reached without assuming the validity of the first Born approximation [16, 30]. In other words the following theorem on broadband invisibility holds.

[8] This means that $v(x)$ is unidirectionally left-invisible for $k = K_n/2 = \pi n/L$ provided that we can neglect terms of order $(\mathfrak{z}_n/k^2)^2$ in the calculation of the reflection and transmission amplitudes.

**Theorem 8** *A scattering potential $v(x)$ is unidirectionally left-invisible for all wavenumbers $k \in \mathbb{R}^+$, if its Fourier transform $\tilde{v}(\mathfrak{K})$ vanishes for all $\mathfrak{K} \leq 0$.*

Because the hypothesis of this theorem is equivalent to the condition that the real and imaginary part of $v(x)$ are connected by the spatial Kramers-Kronig relations, these potentials are sometimes called Kramers-Kronig potentials.[9] It is well-known that they have a power-law decay at spatial infinities.[10]

The unidirectional invisibility of the potential (76) for $k = K/2 = \pi/L$ is a perturbative result [46]; it is violated for sufficiently large values of $|\mathfrak{z}|$, [18, 29]. This potential does however support exact (nonperturbative) unidirectional invisibility for particular values of $\mathfrak{z}$, [51]. Another example of a finite-range potential with exact unidirectional invisibility is (77) with

$$f(x) := \frac{-2\alpha K^2(3 - 2e^{iKx})}{e^{2iKx} + \alpha(1 - e^{iKx})^2},$$

where $\alpha$ and $K$ are real parameters. It turns out that this potential is unidirectionally right-invisible for $k = K/2 = \pi n/L$ with $n$ being any positive integer provided that $\alpha > -1/4$, [47]. The simplest scattering potential supporting exact unidirectional invisibility are barrier potentials of the form $v(x) = \mathfrak{z}_1\chi_{[-a_1,0)} + \mathfrak{z}_2\chi_{[0,a_2]}$ where $\mathfrak{z}_j$ and $a_j$ are respectively complex and positive real parameters [43]. See also [67].

## 5 Spectral Singularities, Resonances, and Bound States

In Sect. 4 we show that the Wronskian of the Jost solutions $\psi_\pm$ of the Schrödinger equation for a scattering potential $v(x)$ is given by

$$W(x) = \frac{2ik}{\mathfrak{t}(k)} = 2ikM_{22}(k). \tag{79}$$

This in particular implies that $\psi_\pm$ are linearly dependent solutions of the Schrödinger equation (2) whenever $k$ is a real and positive zero of $M_{22}(k)$. This represents a physical wavenumber $k$ at which $\mathfrak{t}(k)$ blows up. The corresponding value of the energy, $E := k^2$, which belongs to the continuous spectrum of the Schrödinger operator, $-\frac{d^2}{dx^2} + v(x)$, is called a *spectral singularity*[11] of the potential [38].

[9]For a review of basic properties of these potentials, see [15].

[10]This explains why Theorems 6 and 8 do not conflict.

[11]The notion of a spectral singularity was originally introduced in [60] for Schrödinger operators in the half-line. It was subsequently generalized to the case of full-line in [21]. The term "spectral singularity" was originally used to refer to this notion in [69]. For a readable account of basic mathematical facts about spectral singularities and further references, see [14].

If $k_0^2$ is a spectral singularity, $M_{22}(k_0) = 0$, but because $\det \mathbf{M}(k_0) = 1$, neither of $M_{12}(k_0)$ and $M_{21}(k_0)$ can vanish. In light of (53), this implies that similarly to the transmission amplitude $\mathfrak{t}(k)$, the reflection amplitudes $\mathfrak{r}_{l/r}(k)$ blow up at $k = k_0$. Furthermore, (23) shows that whenever $M_{22}(k) = 0$,

$$\psi_+(x) = -\frac{M_{21}(k)}{M_{12}(k)}\,\psi_-(x) \to \begin{cases} -M_{21}(k)e^{-ikx} & \text{for } x \to -\infty, \\ e^{ikx} & \text{for } x \to +\infty. \end{cases} \tag{80}$$

Application of this relation for $k = k_0$ shows that at a spectral singularity Jost solutions $\psi_\pm(x)$ are scattering solutions of the Schrödinger equation that satisfy outgoing asymptotic boundary conditions. These are also known as the Seigert boundary conditions [70] which provide a standard description of resonances.

Consider a solution $\psi(x)$ of the time-independent Schrödinger equation (2) for a general complex value of the energy $k^2$ and suppose that it satisfies the outgoing asymptotic boundary conditions:

$$\psi(x) \to N_\pm(k)\, e^{\pm ikx} \quad \text{for} \quad x \to \pm\infty, \tag{81}$$

where $N_\pm(k)$ are nonzero complex coefficients. $\psi(x)$ corresponds to a solution $\psi(x,t)$ of the time-dependent Schrödinger equation, $i\partial_t\psi(x,t) = -\partial_x^2\psi(x,t) + v(x)\psi(x,t)$, namely

$$\psi(x,t) := e^{-ik^2t}\psi(x) = e^{-\Gamma t}e^{-iEt}\psi(x), \tag{82}$$

where

$$E := \mathrm{Re}(k)^2 - \mathrm{Im}(k)^2, \qquad\qquad \Gamma := -2\mathrm{Re}(k)\mathrm{Im}(k). \tag{83}$$

If $\Gamma > 0$, $\psi(x,t)$ decays exponentially as $t \to \infty$. In this case, we identify $\psi(x,t)$ with a *resonance*. The quantity $\Gamma$ which determines its decay rate is called the *width* of the resonance. If $\Gamma < 0$, $\psi(x,t)$ grows exponentially as $t \to \infty$, and we call it an *antiresonance*. It is not difficult to see that resonances and antiresonances are also zeros of $M_{22}(k)$. But the corresponding value of $k^2$ lie in the lower and upper complex energy half-planes,

$$\mathcal{E}_{\text{lower}} := \{k^2 \in \mathbb{C} \mid \mathrm{Im}(k^2) < 0\}, \qquad \mathcal{E}_{\text{upper}} := \{k^2 \in \mathbb{C} \mid \mathrm{Im}(k^2) > 0\},$$

respectively.

The Jost solutions of the time-independent Schrödinger equation (2) that correspond to a spectral singularity satisfy the above description of a resonance except that for a spectral singularity $k$ is real. This suggests identifying these solutions with certain *zero-width resonances* [38].[12] Note that spectral singularities lie on the positive real axis in the complex energy plane:

$$\mathcal{E}_+ := \{k^2 \in \mathbb{C} \mid \mathrm{Re}(k^2) > 0 \text{ and } \mathrm{Im}(k^2) = 0\}. \tag{84}$$

There is another way in which we can have a real zero of $M_{22}(k)$ such that $\Gamma = 0$. This is when $k$ is purely imaginary; i.e., $E = k^2 \in \mathbb{R}^-$. Let us set $k = i\sqrt{|E|}$. Then, according to (80), $\psi_+$ determines a solution of the time-independent Schrödinger equation that decays exponentially at spatial infinities. This solution is clearly square-integrable. Therefore its energy $E = k^2$, which is real and negative, belongs to the point spectrum of the Schrödinger operator $-\frac{d^2}{dx^2} + v(x)$; it is a real and negative eigenvalue of this operator that corresponds to a bound state of the potential $v(x)$. If $k$ is a zero of $M_{22}(k)$ that lies in the upper-half $k$-plane, i.e., $\mathrm{Im}(k) > 0$, then $|\psi_+(x)|$ is again exponentially decaying as $x \to \pm\infty$. Therefore $\psi_+(x)$ is a square-integrable function and $k^2$ is a complex eigenvalue of $-\frac{d^2}{dx^2} + v(x)$.

Note that the above discussion of the interpretation of the zeros of $M_{22}(k)$ as spectral singularities, resonances, antiresonances, and eigenvalues of the Schrödinger operator $-\frac{d^2}{dx^2} + v(x)$ applies to any scattering potential. As shown in [21], in this case the Jost solutions $\psi_\pm$ and consequently the entries of the transfer matrix are continuous functions of $k$ for $\mathrm{Im}(k) \geq 0$. They might not however be holomorphic in any region containing the real axis in the complex $k$-plane. If there is such a region in which $M_{22}(k)$ is a nonzero holomorphic function, then the zeros of $M_{22}(k)$ that lie in this region form a discrete isolated set of points. This in turn implies that one cannot have spectral singularities in an extended interval of real numbers other than the whole positive real axis. In particular we have the following result.

**Theorem 9** *If $v(x)$ is a real or complex potential with a finite range or an asymptotic exponential decay, so that (13) holds for some $\mu_\pm \in \mathbb{R}^+$, then either its spectral singularities are isolated points of the positive real axis in the complex energy plane or cover the whole positive real axis.*

Next, we examine the behavior of the eigenvalues $\mathfrak{s}_\pm$ of the $\mathbf{S}$-matrix in the vicinity of a spectral singularity $k_0^2$. As $k \to k_0$, $\epsilon := M_{22}(k)$ tends to zero. Because the entries of the transfer matrix are continuous functions on the upper half-plane and $\mathrm{Im}(k_0) \geq 0$, none of them blow up at $k = k_0$. We also know that $\det \mathbf{M}(k) = 1$. In view of these observations and (50), we can show that the eigenvalues of the $\mathbf{S}$-matrix for a scattering potential satisfy

[12] Spectral singularities must be distinguished with the solutions of the time-independent Schrödinger equation that correspond to a bound state in the continuum [17, 72] for the following reasons: (1) They define scattering states that do not decay at spatial infinities. (2) They may exist for exponentially decaying and short-range potentials. (3) As we explain in Sect. 8, real potentials cannot have spectral singularities. None of these holds for bound states in the continuum.

$$\mathfrak{s}_\pm(k) \to \frac{1}{\epsilon} \pm \frac{1}{|\epsilon|} \mp \frac{\operatorname{sgn}(\epsilon) M_{11}(k_0)}{2} \qquad \text{for} \qquad k \to k_0. \tag{85}$$

This implies that as $k^2$ approaches a spectral singularity, one of the eigenvalues of $\mathbf{S}(k)$ diverges while the other attains a finite limit. More specifically we have the following result.

**Theorem 10** *Let $k_0^2$ be a spectral singularity of a scattering potential $v(x)$. Then as $k \to k_0$ the eigenvalues (50) of the $\mathbf{S}$-matrix behave as follows. Either $\mathfrak{s}_-(k) \to -M_{11}(k_0)/2$ and $|\mathfrak{s}_+(k)| \to \infty$, or $|\mathfrak{s}_-(k)| \to \infty$ and $\mathfrak{s}_+(k) \to M_{11}(k_0)/2$.*

Now, suppose that $v(x)$ is a scattering potential such that $\det \mathbf{S}(k)$ is a bounded function of $k$. Then Theorem 10 implies that $M_{11}(k_0) = 0$ whenever $k_0^2$ is a spectral singularities of $v(k)$, i.e., $k_0$ is a common zero of $M_{11}(k)$ and $M_{22}(k)$. Spectral singularities satisfying this condition are said to be *self-dual* [42]. We study these in Sect. 9.

Let us examine the spectral singularities of a couple of exactly solvable potentials.

First, consider a delta-function potential with a complex coupling constant $\mathfrak{z}$, [37],

$$v(x) = \mathfrak{z}\, \delta(x). \tag{86}$$

We can determine its transfer matrix using (8), (26), and (27) with $c = 0$. This gives

$$\mathbf{M}(k) = \begin{bmatrix} 1 - i\mathfrak{z}/2k & -i\mathfrak{z}/2k \\ i\mathfrak{z}/2k & 1 + i\mathfrak{z}/2k \end{bmatrix}. \tag{87}$$

In view of this relation and (53),

$$\mathfrak{r}_l(k) = \mathfrak{r}_r(k) = \frac{-i\mathfrak{z}}{2k + i\mathfrak{z}}, \qquad \mathfrak{t}(k) = \frac{2k}{2k + i\mathfrak{z}}. \tag{88}$$

The following are consequences of the fact that $M_{22}(k)$ has a single zero, namely $k_0 = -i\mathfrak{z}/2$.

- The delta-function potential has a spectral singularity, if and only if $\mathfrak{z}$ is purely imaginary and $\operatorname{Im}(\mathfrak{z}) > 0$, i.e., $\mathfrak{z} = i\zeta$ for some $\zeta \in \mathbb{R}^+$. In this case, $k_0 = \zeta/2$, the spectral singularity has the value $k_0^2 = \zeta^2/4$, and

$$\psi_+(x) = e^{\pm i k_0 x} \qquad \text{for} \qquad \pm x \geq 0. \tag{89}$$

- It has a single resonance (respectively antiresonance) with a square-integrable position wave function $\psi(x)$ if and only if $\operatorname{Im}(\mathfrak{z}) > 0$ (respectively $< 0$) and $\operatorname{Re}(\mathfrak{z}) < 0$. In this case $\psi(x)$ is a constant multiple of the right-hand side of (89) with $k_0 = [\operatorname{Im}(\mathfrak{z}) - i\operatorname{Re}(\mathfrak{z})]/2$.

- It has a bound state with a real and negative energy if and only if $\mathfrak{z} \in \mathbb{R}^-$. The position wave function for this state is a constant multiple of the right-hand side of (89) with $k_0 = i|\mathfrak{z}|/2$.

Next, we consider the spectral singularities of the complex barrier potential (56). According to (59), zeros $k_0$ of $M_{22}(k)$ satisfy

$$\cos(k_0 L\mathfrak{n}_0) - i\mathfrak{n}_{0+}\sin(k_0 L\mathfrak{n}_0) = 0, \tag{90}$$

where

$$\mathfrak{n}_0 := \sqrt{1 - \frac{\mathfrak{z}}{k_0^2}}, \qquad\qquad \mathfrak{n}_{0+} := \frac{\mathfrak{n}_0^2 + 1}{2\mathfrak{n}_0}. \tag{91}$$

It is not difficult to express (90) in the form:

$$e^{-2ik_0 L\mathfrak{n}_0} = \left(\frac{\mathfrak{n}_0 - 1}{\mathfrak{n}_0 + 1}\right)^2. \tag{92}$$

$k_0^2$ is a spectral singularity if and only if $k_0$ is a positive real number satisfying this relation. For such a $k_0$, we can write (92) as a pair of real equations for the $k_0$, $\eta_0 := \mathrm{Re}(\mathfrak{n}_0)$, and $\kappa_0 := \mathrm{Im}(\mathfrak{n}_0)$. Because

$$\mathfrak{n}_0 = \eta_0 + i\kappa_0, \tag{93}$$

evaluating the modulus of both side of (92) we find

$$\kappa_0 = \frac{1}{2k_0 L}\ln\left|\frac{(\eta_0 - 1)^2 + \kappa_0^2}{(\eta_0 + 1)^2 + \kappa_0^2}\right|. \tag{94}$$

Similarly, equating the phase angles of both side of (92), we obtain

$$k_0 = \frac{2\pi m - \varphi_0}{2L\eta_0}, \tag{95}$$

where $m$ is a positive integer, and $\varphi_0$ is the principle argument of the right-hand side of (92), i.e.,

$$\varphi_0 = \begin{cases} \arctan(\alpha_0) & \text{for } \eta_0^2 + \kappa_0^2 \geq 1, \\ \arctan(\alpha_0) - \pi & \text{for } \eta_0^2 + \kappa_0^2 < 1, \end{cases} \qquad \alpha_0 := \frac{2\kappa_0}{(\eta_0^2 + 1)^2 + \kappa_0^2}. \tag{96}$$

Next, let us identify the barrier potential (56) with an optical potential (4) that describes the scattering of normally incident polarized electromagnetic waves by an infinite slab of homogeneous nonmagnetic material. We choose a coordinate system in which the slab occupies the space confined between the planes $x = 0$ and $x = L$, and the wave is polarized along the $y$-direction and propagates along the $x$-direction. Then the relative permittivity of the system that enters the Helmholtz equation (3) has the form:

$$\hat{\varepsilon}(x) = \begin{cases} \hat{\varepsilon}_{\rm slab} & \text{for } x \in [0, L], \\ 1 & \text{for } x \notin [0, L], \end{cases} \tag{97}$$

where $\hat{\varepsilon}_{\rm slab}$ is the relative permittivity of the slab. In general this takes a possibly complex constant value. We can identify the Helmholtz equation with the Schrödinger equation (2) provided that $v(x)$ is the barrier potential (56) with $\mathfrak{z} = k^2(1 - \hat{\varepsilon}_{\rm slab})$. Substituting this equation in (58), we find $\mathfrak{n} = \sqrt{\hat{\epsilon}}$. Therefore $\mathfrak{n}$ is the refractive index of the slab.

According to (94) the optical system we have described has a spectral singularity, if the imaginary part of the refractive index of our slab is negative. This is precisely the case where the slab is made out of gain material. To see this we note that the gain coefficient of a homogeneous medium is related to its refractive index according to

$$g = -\frac{4\pi\,{\rm Im}(\mathfrak{n})}{\lambda} = -2k{\rm Im}(\mathfrak{n}), \tag{98}$$

where $\lambda = 2\pi/k$ is the wavelength [71]. If the refractive index of the slab equals $\mathfrak{n}_0$, it emits coherent outgoing radiation of wavelength $\lambda_0 = 2\pi/k_0$, i.e., it acts as a laser. In view of (94), for $k = k_0$ and $\mathfrak{n} = \mathfrak{n}_0$, the gain coefficient (98) is given by [40]:

$$g = \frac{1}{L}\ln\left|\frac{(\eta_0+1)^2+\kappa_0^2}{(\eta_0-1)^2+\kappa_0^2}\right| = \frac{2}{L}\ln\left|\frac{\mathfrak{n}_0+1}{\mathfrak{n}_0-1}\right|. \tag{99}$$

This relation is known as the *laser threshold condition* in optics [71]. It is usually derived by balancing the energy input of the laser by the sum of its energy output and losses. Here we obtain it using the notion of spectral singularity, i.e., demanding the existence of purely outgoing solutions of the wave equation. Notice that this condition also yields a formula for the available laser modes, namely (95). For typical lasers, $k_0 L \gg 1$. This implies $m \gg 1$ which together with (95) give $k_0 \approx \pi m/L{\rm Re}(\mathfrak{n}_0)$. The latter is also a well-known result in optics.

The notion of spectral singularity can be extended to more general scattering problems. This is done by identifying it with the values of $k^2$ at which the left or right reflection and transmission coefficients blow up. This corresponds to situations where $\psi(x)$ satisfies purely outgoing boundary conditions.[13] For a linear scattering problem, the assumption $\det\mathbf{M}(k) \neq 0$ together with Eqs. (22) imply that spectral

[13]The importance of purely outgoing waves in the laser theory predates the discovery of their connection to the mathematics of spectral singularities. See for example [73].

singularities are given by the real and positive zeros of $M_{22}(k)$ and that they are always bidirectional, i.e., both the left and right reflection and transmission coefficients diverge at a spectral singularity.

Determination of spectral singularities of an optical system having an arbitrary geometry is equivalent to finding its laser threshold condition. This observation has been employed for obtaining laser threshold condition for bilayer [42], cylindrical [57], and spherical [55, 56, 58] lasers. A brief review of the physical aspects of spectral singularities is provided in [49]. For a discussion of the spectral singularities of nonlinear Schrödinger equation and their applications in optics, see [9, 12, 26, 44].

## 6 Space Reflections and Time-Reversal Transformation

In this section we explore the space reflection and time-reversal transformations in quantum mechanics. This requires the knowledge of unitary and Hermitian operators acting in a Hilbert space. Because a precise definition of a Hermitian operator involves certain notions of functional analysis that are not familiar to most physicists, here we provide a less rigorous description. The interested reader may consult [39, 62] for a more careful treatment of the subject.

Consider a linear operator $L$ acting in a Hilbert space $\mathscr{H}$, and let $\prec\cdot,\cdot\succ$ denote the inner product of $\mathscr{H}$. Then the *adjoint* of $L$ is the operator $L^\dagger : \mathscr{H} \to \mathscr{H}$ that satisfies

$$\prec\cdot, L\cdot\succ=\prec L^\dagger\cdot,\cdot\succ.$$

We call $L$ *Hermitian* or *self-adjoint* if $L^\dagger = L$. We call it a *unitary operator* if its domain is $\mathscr{H}$, it is one-to-one and onto, and $L^{-1} = L^\dagger$. These conditions are equivalent to the requirement that

$$\prec L\phi_1, L\phi_2\succ=\prec\phi_1,\phi_2\succ,$$

i.e., $L$ leaves the inner product invariant. Here and in what follows $\phi_1$ and $\phi_2$ are arbitrary elements of $\mathscr{H}$. It turns out that $L$ is unitary if and only if it preserves the norm of the vectors; $\parallel L\phi_1\parallel=\parallel\phi_1\parallel$ where $\parallel\phi_1\parallel:=\sqrt{\prec\phi_1,\phi_1\succ}$.

In the standard quantum mechanical description of the nonrelativistic motion of a particle on a straight line, we take $\mathscr{H}$ to be the space of square integrable functions $L^2(\mathbb{R})$ endowed with the inner product: $\langle\phi_1|\phi_2\rangle := \int_{-\infty}^{\infty}\phi_1(x)^*\phi_2(x)dx$.

Hermitian operators play a basic role in both kinematical and dynamical aspects of quantum mechanics. Observables of quantum systems are described by Hermitian operators not just because they have a real spectrum, but more importantly because their expectation values are real. *Non-Hermitian operators may have a real spectrum and even a complete set of eigenvectors forming a basis of the Hilbert space, but* *there are always states in which their expectation value is not real.*[14] Because the calculation of expectation values involves the inner product of the Hilbert space, a non-Hermitian operator can play the role of an observable of a quantum system, only if we can modify the inner product on the space of state vectors or even the space of state vectors itself [45], so that the operator acts in the new Hilbert space as a Hermitian operator.[15] This leads to different representations of quantum mechanics whose structure is identical to the standard representation that we employ here [39]. The Hamiltonian operator is required to be Hermitian not only because it is usually identified with the energy observable, but also because it ensures the unitarity of time-evolution, i.e., the time-evolution operator defined by the Hamiltonian is a unitary operator. A celebrated result of functional analysis, known as Stone's theorem [64], establishes the converse of this statement. Therefore, *the unitarity of dynamics implies the Hermiticity of the Hamiltonian.* This result also disqualifies non-Hermitian operators from serving as the Hamiltonian operator for a unitary quantum system.

Non-Hermitian operators can nevertheless be employed in the study of open quantum systems and a variety of problems in the areas where some of the axioms of quantum mechanics are violated. This has actually turned out to be more fruitful than the attempts to use non-Hermitian operators for invoking the nonstandard representations of quantum mechanics.

Having reviewed the meaning of Hermiticity and unitarity of an operator and their role in quantum mechanics, we return to the study of space reflections and time-reversal transformation.

For each $a \in \mathbb{R}$, the active transformation, $x \to 2a - x$, corresponds to the reflection of the real line about the point $a$. This transformation induces a mapping of the wave functions $\phi(x)$ according to $\phi(x) \to \phi(2a - x)$. We identify this with the action of a linear operator $\mathcal{P}_a$ in $L^2(\mathbb{R})$, namely $\phi \to \widetilde{\phi} := \mathcal{P}_a\phi$, where

$$(\mathcal{P}_a\phi)(x) := \phi(2a - x). \tag{100}$$

It is easy to show that $\mathcal{P}_a$ is a Hermitian operator. It is also clear that $\mathcal{P}_a^2 = I$, so that $\mathcal{P}_a^{-1} = \mathcal{P}_a$. Combining this with the Hermiticity of $\mathcal{P}_a$ we conclude that $\mathcal{P}_a$ is also a unitary operator.

We can use $\mathcal{P}_a$ to transform linear operators $L(t)$ acting in $L^2(\mathbb{R})$ according to

$$L(t) \to \widetilde{L}(t) := \mathcal{P}_a\, L(t)\, \mathcal{P}_a^{-1} = \mathcal{P}_a\, L(t)\, \mathcal{P}_a. \tag{101}$$

[14] For a proof of this statement see [39, Appendix]. A more detailed discussion is provided in [68].

[15] This is obviously not always possible. A sufficient condition for the existence of such a modified inner product is that the operator $L$ satisfies the pseudo-Hermiticity relation $L^\dagger = \eta\, L \eta^{-1}$ for a positive-definite bounded linear operator $\eta$ with a bounded inverse. For further discussion of these and related issues see [39, 45] and references therein.

For example, let $\hat{x}$, $\hat{p}$, and $H(t)$ be respectively the standard position, momentum, and Hamiltonian operators acting in $L^2(\mathbb{R})$, i.e.,

$$\hat{x}\,\phi(x) := x\phi(x), \qquad \hat{p}\,\phi(x) := -i\phi'(x), \qquad H(t) = \frac{\hat{p}^2}{2m} + v(\hat{x}, t). \tag{102}$$

We can use (100) to show that

$$\{\hat{x}, \mathcal{P}_a\} = 2aI, \qquad\qquad \{\hat{p}, \mathcal{P}_a\} = 0, \tag{103}$$

where $\{\cdot\,,\,\cdot\}$ stands for the anticommutator of operators. Equations (101), (102), and (103) imply

$$\widetilde{\hat{x}} = 2aI - \hat{x}, \qquad \widetilde{\hat{p}} = -\hat{p}, \qquad \widetilde{H}(t) = \frac{\hat{p}^2}{2m} + v(2aI - \hat{x}, t). \tag{104}$$

The first of these relations justifies the name "space reflection" or "parity operator with respect to $a$" for $\mathcal{P}_a$.

If $H(t)$ is the Hamiltonian operator for a quantum system $\mathcal{S}$, we call the quantum system defined by $\widetilde{H}$ the "space reflection of $\mathcal{S}$ with respect to $a$." Equation (101) and the unitarity of $\mathcal{P}_a$ imply that $\widetilde{H}(t)$ is Hermitian if and only if so is $H(t)$. This means that space reflections of a unitary quantum system are unitary.

An operator $L(t)$ is called *parity-invariant with respect to a* if $\widetilde{L}(t) = L(t)$. In particular, a standard Hamiltonian operator (102) is parity-invariant with respect to $a$ if and only if $v(2a - x, t) = v(x, t)$.

The parity operators $\mathcal{P}_a$ can be generated from $\mathcal{P}_0$ using the space-translation operator $T_a := e^{-ia\hat{p}}$ which satisfies:

$$(T_a\,\phi)(x) = \phi(x - a). \tag{105}$$

To see this we use (100) and (105) to show that

$$(\mathcal{P}_a\phi)(x) = \phi(2a - x) = (\mathcal{P}_0\phi)(x - 2a) = (T_{2a}\mathcal{P}_0\phi)(x).$$

Therefore,

$$\mathcal{P}_a = T_{2a}\mathcal{P}_0. \tag{106}$$

We use the symbol $\mathcal{P}$ for $\mathcal{P}_0$ and refer to it as the *parity operator* in $L^2(\mathbb{R})$. According to this terminology a standard Hamiltonian operator (102) is *parity-invariant* or *$\mathcal{P}$-symmetric* if and only if $v(x, t) = v(-x, t)$. For a time-independent potential $v(x)$, this means that it is an even function.

Next, consider the operation of complex-conjugation of complex-valued functions, $\phi(x) \to \phi(x)^*$. This defines a function $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ according to $(\mathcal{T}\phi)(x) := \phi(x)^*$. Because for any pair of complex numbers $\alpha_1$ and $\alpha_2$,

$$\mathcal{T}(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1^*\mathcal{T}\phi_1 + \alpha_2^*\mathcal{T}\phi_2,$$

$\mathcal{T}$ is an *antilinear operator*. It is also clear that $\mathcal{T}$ squares to the identity operator $I$. In particular, it is invertible, and $\mathcal{T}^{-1} = \mathcal{T}$.

Let us apply $\mathcal{T}$ to both sides of the time-dependent Schrödinger equation,

$$i\frac{d}{dt}\psi(x,t) = H(t)\psi(x,t). \tag{107}$$

This gives $-i\frac{d}{dt}\mathcal{T}\psi(x,t) = \mathcal{T}H(t)\psi(x,t)$. We can write this equation in the form

$$i\frac{d}{d(-t)}\mathcal{T}\psi(x,t) = \overline{H}(-t)\mathcal{T}\psi(x,t), \tag{108}$$

where for a time-dependent linear operator $L(t)$,

$$\overline{L}(t) := \mathcal{T}L(-t)\mathcal{T}^{-1} = \mathcal{T}L(-t)\mathcal{T}. \tag{109}$$

If we make the change of variables:

$$t \to \bar{t} := -t, \qquad \psi(x,t) \to \overline{\psi}(x,\bar{t}) := \psi(x,t)^* = (\mathcal{T}\psi)(x,t),$$

Eq. (108) takes the form $i\frac{d}{d\bar{t}}\overline{\psi}(x,\bar{t}) = \overline{H}(\bar{t}))\overline{\psi}(x,\bar{t})$. Because $t$ and $\bar{t}$ take arbitrary real values, this equation is equivalent to

$$i\frac{d}{dt}\overline{\psi}(x,t) = \overline{H}(t)\overline{\psi}(x,t). \tag{110}$$

We can express the solutions of (107) and (110) in terms of the time-evolution operators $U(t)$ and $\overline{U}(t)$ for the Hamiltonians $H(t)$ and $\overline{H}(t)$. For a given initial state vector $\psi_0(x)$, we have

$$\psi(x,t) = U(t)\psi_0(x), \qquad \overline{\psi}(x,t) = \overline{U}(t)\psi_0(x)^*. \tag{111}$$

According to these relations, as we increase the value of the time label $t$ starting from $t = 0$, the evolution operators $U(t)$ and $\overline{U}(t)$ respectively determine $\psi(x,t)$ and $\overline{\psi}(x,t)$ for $t > 0$. In view of the fact that $\psi(x,-t) = \overline{\psi}(x,t)^*$, we can say that $\overline{U}(t)$ determines $\psi(x,t)$ for $t < 0$. For this reason, the systems described by the Hamiltonian operators $H(t)$ and $\overline{H}(t)$ are said to be the time-reversal of one another. This, in particular, suggests identifying the antilinear operator $\mathcal{T}$ with the *time-reversal operator*.

The above argument leaves a crucial question unanswered: Suppose that $H(t)$ is a Hermitian operator so that it determines a unitary quantum system. Does this imply that the time-reversed system is also unitary? Equivalently, is $\overline{H}(t)$ Hermitian? The answer turns out to be in the affirmative, because $\mathcal{T}$ satisfies

$$\langle\mathcal{T}\phi_1|\mathcal{T}\phi_2\rangle = \int_{-\infty}^{\infty}[\mathcal{T}\phi_1(x)]^*\mathcal{T}\phi_2(x)dx = \int_{-\infty}^{\infty}\phi_1(x)\phi_2(x)^*dx = \langle\phi_2|\phi_1\rangle. \tag{112}$$

With the help of this relation and the Hermiticity of $H(t)$, we can show that

$$\begin{aligned}\langle\phi_1|\overline{H}(t)\phi_2\rangle &= \langle\mathcal{T}^2\phi_1|\overline{H}(t)\phi_2\rangle = \langle\mathcal{T}^2\phi_1|\mathcal{T}H(-t)\mathcal{T}\phi_2\rangle = \langle H(-t)\mathcal{T}\phi_2|\mathcal{T}\phi_1\rangle\\ &= \langle\mathcal{T}\phi_2|H(-t)\mathcal{T}\phi_1\rangle = \langle\mathcal{T}\phi_2|\mathcal{T}^2H(-t)\mathcal{T}\phi_1\rangle = \langle\mathcal{T}H(-t)\mathcal{T}\phi_1|\phi_2\rangle\\ &= \langle\overline{H}(t)\phi_1|\phi_2\rangle.\end{aligned}$$

This concludes the proof of the Hermiticity of $\overline{H}(t)$.

An antilinear operator $\mathfrak{S}$, which by definition satisfies

$$\mathfrak{S}(\alpha_1\phi_1+\alpha_2\phi_2) = \alpha_1^*\mathfrak{S}\phi_1+\alpha_2^*\mathfrak{S}\phi_2,$$

is said to be unitary, if

$$\langle\mathfrak{S}\phi_1|\mathfrak{S}\phi_2\rangle = \langle\phi_2|\phi_1\rangle. \tag{113}$$

Unitary antilinear operators are also called "*antiunitary operators*" [76]. Similarly to unitary linear operators they preserve the norm of state vectors.

Equation (112) means that $\mathcal{T}$ is an antiunitary operator. There are other antiunitary operators that square to identity and share the time-reversal property of $\mathcal{T}$.[16] This implies that in general $\mathcal{T}$ is not the only possible choice for a time-reversal operator [35]. In what follows, however, we take $\mathcal{T}$ to implement the time-reversal transformation in $L^2(\mathbb{R})$ and refer to it as the *time-reversal operator*.

A possibly time-dependent linear operator $L(t)$ is said to be *time-reversal-invariat* or *real* if $\overline{L}(t) = L(t)$. It is called an *imaginary operator* if $\overline{L}(t) = -L(t)$. For example, the standard position operator $\hat{x}$ is real, because

$$\bar{\hat{x}}\,\phi(x) = \mathcal{T}\hat{x}\mathcal{T}\phi(x) = [x\phi(x)^*]^* = x\phi(x) = \hat{x}\,\phi(x),$$

while the standard momentum operator $\hat{p}$ is imaginary, because

$$\bar{\hat{p}}\,\phi(x)=\mathcal{T}\hat{p}\mathcal{T}\phi(x)=\mathcal{T}\left[-i\frac{d}{dx}\phi(x)^*\right]=i\mathcal{T}\left[\frac{d}{dx}\phi(x)^*\right]=i\frac{d}{dx}\phi(x)=-\,\hat{p}\phi(x).$$

[16] A simple examples is $\mathcal{T}_\tau := e^{i\tau}\mathcal{T}$ where $\tau\in\mathbb{R}$.

Clearly $L(t)$ is an imaginary operator if and only if $iL(x)$ is real. In particular, $iI$ is imaginary, because $I$ is a real operator. Note also that time-independent linear operators $L_R$ and $L_I$ are respectively real and imaginary if and only if

$$[L_R, \mathcal{T}] = 0, \qquad\qquad \{L_I, \mathcal{T}\} = 0.$$

We can easily show that the real multiples, sums, and products of real operators are real. This for instance implies that $\hat{p}^2 = -(i\hat{p})^2$ is a real operator. In light of this observation, the time-reversal of a standard Hamiltonian operator (102) is given by $\overline{H}(t) = \frac{\hat{p}^2}{2m} + \overline{v(\hat{x}, t)}$, where

$$\overline{v(\hat{x}, t)}\phi(x) = \mathcal{T}v(\hat{x}, -t)\mathcal{T}\phi(x) = \mathcal{T}[v(x, -t)\phi(x)^*] = v(x, -t)^*\phi(x).$$

This shows that $v(\hat{x}, t)$ is a real operator provided that $v(x, -t) = v(x, t)^*$. In particular, for a time-independent standard Hamiltonian,

$$H = \frac{\hat{p}^2}{2m} + v(\hat{x}), \tag{114}$$

we have

$$\overline{H} = \frac{\hat{p}^2}{2m} + v(\hat{x})^*, \tag{115}$$

where $v(\hat{x})^*\phi(x) := \overline{v(x)}\phi(x) = v(x)^*\phi(x)$. The Hamiltonian (114) is therefore real if and only if $v(x)$ is a real-valued potential.

Next, we explore the consequences of the combined action of parity and time-reversal transformations. This is realized in $L^2(\mathbb{R})$ by $\mathcal{PT}$ whose effect on the wave functions $\phi(x)$ and time-dependent linear operators $L(t)$ are give by

$$\begin{aligned}
\phi(x) &\longrightarrow \tilde{\overline{\phi}}(x) := (\mathcal{PT}\phi)(x) = \phi(-x, t)^*,\\
L(t) &\longrightarrow \tilde{\overline{L}}(t) := \mathcal{P}\left[\mathcal{T}L(-t)\mathcal{T}^{-1}\right]\mathcal{P}^{-1} = \mathcal{PT}L(-t)(\mathcal{PT})^{-1} = \mathcal{PT}L(-t)\mathcal{PT}.
\end{aligned}$$

Here, in the last equality we have used the fact that $\mathcal{P}$ and $\mathcal{T}$ commute and square to identity;

$$[\mathcal{P}, \mathcal{T}] = 0, \qquad\qquad \mathcal{P}^2 = \mathcal{T}^2 = I. \tag{116}$$

Because $\mathcal{P}$ and $\mathcal{T}$ are respectively unitary and antiunitary operators,

$$\begin{aligned}
&\mathcal{PT}(\alpha_1\phi_1 + \alpha_1\phi_2) = \mathcal{P}(\alpha^*\mathcal{T}\phi_1 + \alpha_2^*\mathcal{T}\phi_2) = \alpha^*\mathcal{PT}\phi_1 + \alpha_2^*\mathcal{PT}\phi_2,\\
&\langle\mathcal{PT}\phi_1|\mathcal{PT}\phi_2\rangle = \langle\mathcal{T}\phi_1|\mathcal{T}\phi_2\rangle = \langle\phi_2|\phi_1\rangle.
\end{aligned}$$

These show that $\mathcal{PT}$ is an antiunitary operator. The same is true about $\mathcal{P}_a\mathcal{T}$.

We can use (116) and

$$\bar{\hat{x}} = \mathcal{T}\,\hat{x}\,\mathcal{T}^{-1} = \hat{x}, \qquad \bar{\hat{p}} = \mathcal{T}\,\hat{p}\,\mathcal{T}^{-1} = -\hat{p},$$
$$\tilde{\hat{x}} = \mathcal{P}\,\hat{x}\,\mathcal{P}^{-1} = -\hat{x}, \qquad \tilde{\hat{p}} = \mathcal{P}\,\hat{p}\,\mathcal{P}^{-1} = -\hat{p}, \tag{117}$$

to show that

$$\tilde{\bar{\hat{x}}} = \mathcal{PT}\,\hat{x}\,(\mathcal{PT})^{-1} = -\hat{x}, \qquad \tilde{\bar{\hat{p}}} = \mathcal{PT}\,\hat{p}\,(\mathcal{PT})^{-1} = \hat{p}. \tag{118}$$

In other words,

$$\{\hat{x}, \mathcal{PT}\} = 0, \qquad [\,\hat{p}, \mathcal{PT}] = 0. \tag{119}$$

Another consequence of (118) and the antilinearity of $\mathcal{PT}$ is that it transforms a standard Hamiltonian operator of the form (114) to

$$\tilde{\bar{H}} = \frac{\hat{p}^2}{2m} + v(-\hat{x})^*. \tag{120}$$

A linear operator $L(t)$ is said to be $\mathcal{PT}$*-symmetric* if it is invariant under the combined action of $\mathcal{P}$ and $\mathcal{T}$, i.e., $L(t) \to \tilde{\bar{L}}(t) = L(t)$. For a time-independent operator $L$, this means

$$[L, \mathcal{PT}] = 0.$$

In particular, $\hat{p}$ is $\mathcal{PT}$-symmetric, and a time-independent standard Hamiltonian $H$ is $\mathcal{PT}$-symmetric if and only if its potential is $\mathcal{PT}$-symmetric, i.e., $v(-x)^* = v(x)$. In terms of the real and imaginary parts of $v(x)$, which we denote by $v_r(x)$ and $v_i(x)$, this condition takes the form

$$v_r(-x) = v_r(x), \qquad v_i(-x) = -v_i(x). \tag{121}$$

Therefore, the real and imaginary parts of a $\mathcal{PT}$-symmetric potential are respectively even and odd functions. Similarly, it follows that $H$ is $\mathcal{P}_a\mathcal{T}$-symmetric if and only if $v(2a - x)^* = v(x)$. This is equivalent to

$$v_r(2a - x) = v_r(x), \qquad v_i(2a - x) = -v_i(x). \tag{122}$$

## 7 $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-Transformation of the Scattering Data

Consider the scattering problem for a wave equation in one dimension that admits solutions $\psi(x)$ satisfying the asymptotic boundary conditions (1). Suppose that for $x \to \pm\infty$ the parity, time-reversal, and space translations respectively transform $\psi(x)$ according to:

$$\begin{aligned}
&\psi(x)\xrightarrow{\mathcal{P}}\widetilde{\psi}(x):=\psi(-x), &&\psi(x)\xrightarrow{\mathcal{T}}\overline{\psi}(x):=\psi(x)^*,\\
&\psi(x)\xrightarrow{\mathcal{T}_a}\psi_a(x):=\psi(x-a).
\end{aligned}\tag{123}$$

It is easy to see that these transformations leave the asymptotic boundary conditions (1) form-invariant. This shows that the transformed wave functions, $\widetilde{\psi}(x)$, $\overline{\psi}(x)$, and $\psi_a(x)$ also define consistent scattering problems. We wish to explore the behaviour of the corresponding reflection and transmission amplitudes. To do this, we confine our attention to situations where we can define a transfer matrix $\mathbf{M}(k)$ and examine the effect of the transformations (123) on $\mathbf{M}(k)$.

Let $\widetilde{\mathbf{M}}(k)$, $\overline{\mathbf{M}}(k)$, and $\mathbf{M}_a(k)$ respectively denote the transfer matrix for $\widetilde{\psi}(x)$, $\overline{\psi}(x)$, and $\psi_a(x)$. We can use (1), (14), and (123) to relate them to $\mathbf{M}(k)$. This requires expressing the asymptotic expression for $\widetilde{\psi}(x)$, $\overline{\psi}(x)$, and $\psi_a(x)$ in the form (1) with $(A_\pm, B_\pm)$ respectively replaced by $(\widetilde{A}_\pm, \widetilde{B}_\pm)$, $(\overline{A}_\pm, \overline{B}_\pm)$, and $(A_{a\pm}, B_{a\pm})$. In this way we find asymptotic formulas for $\widetilde{\psi}(x)$, $\overline{\psi}(x)$ and $\psi_a(x)$ that together with (123) imply:

$$\widetilde{A}_\pm = B_\mp, \qquad \widetilde{B}_\pm = A_\mp, \tag{124}$$

$$\overline{A}_\pm = B_\pm^*, \qquad \overline{B}_\pm = A_\pm^*, \tag{125}$$

$$A_{a\pm} = e^{-iak}A_\pm, \qquad B_{a\pm} = e^{iak}B_\pm. \tag{126}$$

Recalling that the transfer matrices $\widetilde{\mathbf{M}}$, $\overline{\mathbf{M}}$, and $\mathbf{M}_a$ satisfy

$$\begin{bmatrix}\widetilde{A}_+\\ \widetilde{B}_+\end{bmatrix}=\widetilde{\mathbf{M}}\begin{bmatrix}\widetilde{A}_-\\ \widetilde{B}_-\end{bmatrix},\quad \begin{bmatrix}\overline{A}_+\\ \overline{B}_+\end{bmatrix}=\overline{\mathbf{M}}\begin{bmatrix}\overline{A}_-\\ \overline{B}_-\end{bmatrix},\quad \begin{bmatrix}A_{a+}\\ B_{a+}\end{bmatrix}=\mathbf{M}_a\begin{bmatrix}A_{a-}\\ B_{a-}\end{bmatrix}, \tag{127}$$

we can use (14), (124), (125), and (126) to infer:

$$\widetilde{\mathbf{M}}=\boldsymbol{\sigma}_1\mathbf{M}^{-1}\boldsymbol{\sigma}_1,\qquad \overline{\mathbf{M}}=\boldsymbol{\sigma}_1\mathbf{M}^*\boldsymbol{\sigma}_1,\qquad \mathbf{M}_a=e^{-iak\boldsymbol{\sigma}_3}\mathbf{M}\,e^{iak\boldsymbol{\sigma}_3}, \tag{128}$$

where

$$\boldsymbol{\sigma}_1:=\begin{bmatrix}0&1\\1&0\end{bmatrix},\qquad \boldsymbol{\sigma}_3:=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\qquad e^{ia\boldsymbol{\sigma}_3}=\begin{bmatrix}e^{ia}&0\\0&e^{-ia}\end{bmatrix}.$$

It is instructive to examine the explicit expression for the entries of $\widetilde{\mathbf{M}}$, $\overline{\mathbf{M}}$, and $\mathbf{M}_a$ . According to (128), they have the form:

$$\widetilde{M}_{11}=\frac{M_{11}}{\det\mathbf{M}},\quad \widetilde{M}_{12}=-\frac{M_{21}}{\det\mathbf{M}},\quad \widetilde{M}_{21}=-\frac{M_{12}}{\det\mathbf{M}},\quad \widetilde{M}_{22}=\frac{M_{22}}{\det\mathbf{M}}, \tag{129}$$

$$\overline{M}_{11}=M_{22}^*,\quad \overline{M}_{12}=M_{21}^*,\quad \overline{M}_{21}=M_{12}^*,\quad \overline{M}_{22}=M_{11}^*, \tag{130}$$

$$M_{a11}=M_{11},\quad M_{a12}=e^{-2iak}M_{12},\quad M_{a21}=e^{2iak}M_{21},\quad M_{a22}=M_{22}. \tag{131}$$

We can use these relations together with (22) to compute the reflection and transmission amplitudes for the reflected, time-reversed, and translated waves, $\widetilde{\psi}(x)$, $\overline{\psi}(x)$, and $\psi_a(x)$. These are respectively given by

$$\widetilde{\mathfrak{r}}_l=\mathfrak{r}_r,\qquad \widetilde{\mathfrak{t}}_l=\mathfrak{t}_r,\qquad \widetilde{\mathfrak{r}}_r=\mathfrak{r}_l,\qquad \widetilde{\mathfrak{t}}_r=\mathfrak{t}_l, \tag{132}$$

$$\bar{\mathfrak{r}}_l=-\frac{\mathfrak{r}_r^*}{\mathfrak{D}^*},\qquad \bar{\mathfrak{t}}_l=\frac{\mathfrak{t}_l^*}{\mathfrak{D}^*},\qquad \bar{\mathfrak{r}}_r=-\frac{\mathfrak{r}_l^*}{\mathfrak{D}^*},\qquad \bar{\mathfrak{t}}_r=\frac{\mathfrak{t}_r^*}{\mathfrak{D}^*}, \tag{133}$$

$$\mathfrak{r}_{al}=e^{2iak}\,\mathfrak{r}_l,\qquad \mathfrak{t}_{al}=\mathfrak{t}_l\qquad \mathfrak{r}_{ar}=e^{-2iak}\,\mathfrak{r}_r,\qquad \mathfrak{t}_{ar}=\mathfrak{t}_r, \tag{134}$$

where we recall that $\mathfrak{D}:=M_{11}/M_{22}=\mathfrak{t}_l\mathfrak{t}_r-\mathfrak{r}_l\mathfrak{r}_r=\det\mathbf{S}$.

Next, we examine the effect of $\mathcal{P}_a$ on the scattering data. Because in view of (106) we have $\mathcal{P}_a=T_{2a}\mathcal{P}$, $\mathcal{P}_a$ transforms the transfer matrix $\mathbf{M}$ according to

$$\mathbf{M}\xrightarrow{\mathcal{P}_a}\widetilde{M}_{2a}=e^{-i2ak\boldsymbol{\sigma}_3}\boldsymbol{\sigma}_1\mathbf{M}^{-1}\boldsymbol{\sigma}_1e^{i2ak\boldsymbol{\sigma}_3}=\frac{1}{\det\mathbf{M}}\begin{bmatrix}M_{11} & -e^{-4iak}M_{21}\\ -e^{4aik}M_{12} & M_{22}\end{bmatrix}. \tag{135}$$

Here we have made use of (128) and the identity

$$e^{-i\varphi\boldsymbol{\sigma}_3}\boldsymbol{\sigma}_1=\boldsymbol{\sigma}_1e^{i\varphi\boldsymbol{\sigma}_3}=\begin{bmatrix}0 & e^{-i\varphi}\\ e^{i\varphi} & 0\end{bmatrix}.$$

Using (22) and (135), we obtain

$$\mathfrak{r}_l\xrightarrow{\mathcal{P}_a}e^{4iak}\mathfrak{r}_r=e^{4iak}\widetilde{\mathfrak{r}}_l,\qquad \mathfrak{t}_l\xrightarrow{\mathcal{P}_a}\mathfrak{t}_r=\widetilde{\mathfrak{t}}_r, \tag{136}$$

$$\mathfrak{r}_r\xrightarrow{\mathcal{P}_a}e^{-4iak}\mathfrak{r}_l=e^{-4iak}\widetilde{\mathfrak{r}}_r,\qquad \mathfrak{t}_r\xrightarrow{\mathcal{P}_a}\mathfrak{t}_l=\widetilde{\mathfrak{t}}_r. \tag{137}$$

These equations show that the effect of a space reflection about a point $a\neq 0$ introduces the extra phase factors $e^{\pm 4iak}$ in the expression for the $\mathcal{P}$-transformed reflection amplitudes. In particular, it does not affect the zeros and singularities of the reflection and transmission amplitudes of the system.

We now study the implication of $\mathcal{PT}$ on the scattering data. According to (128), the $\mathcal{PT}$-transformation of the transfer matrix $\mathbf{M}(t)$ yields

$$\begin{aligned}\mathbf{M}\xrightarrow{\mathcal{PT}}\widetilde{\overline{\mathbf{M}}}&=\boldsymbol{\sigma}_1[\boldsymbol{\sigma}_1\mathbf{M}^*\boldsymbol{\sigma}_1]^{-1}\boldsymbol{\sigma}_1=\mathbf{M}^{-1*}\\ &=\frac{1}{\det\mathbf{M}^*}\begin{bmatrix}M_{22}^* & -M_{12}^*\\ -M_{21}^* & M_{11}^*\end{bmatrix}.\end{aligned} \tag{138}$$

In particular,

$$\det \mathbf{M} \xrightarrow{\mathcal{PT}} \det \tilde{\tilde{\mathbf{M}}} = \frac{1}{\det \mathbf{M}^*}, \tag{139}$$

$$M_{11} \xrightarrow{\mathcal{PT}} \tilde{\tilde{M}}_{11} := \frac{M_{22}^*}{\det \mathbf{M}^*}, \qquad M_{12} \xrightarrow{\mathcal{PT}} \tilde{\tilde{M}}_{12} := -\frac{M_{12}^*}{\det \mathbf{M}^*}, \tag{140}$$

$$M_{21} \xrightarrow{\mathcal{PT}} \tilde{\tilde{M}}_{21} := -\frac{M_{21}^*}{\det \mathbf{M}^*}, \qquad M_{22} \xrightarrow{\mathcal{PT}} \tilde{\tilde{M}}_{22} := \frac{M_{11}^*}{\det \mathbf{M}^*}. \tag{141}$$

With the help of these relations and (22) or alternatively (132) and (133), we can derive the following expressions for the $\mathcal{PT}$-transformed reflection and transmission amplitudes.

$$\tilde{\tilde{\mathfrak{r}}}_l = -\frac{\mathfrak{r}_l^*}{\mathfrak{D}^*}, \qquad \tilde{\tilde{\mathfrak{t}}}_l = \frac{\mathfrak{t}_r^*}{\mathfrak{D}^*}, \qquad \tilde{\tilde{\mathfrak{r}}}_r = -\frac{\mathfrak{r}_r^*}{\mathfrak{D}^*}, \qquad \tilde{\tilde{\mathfrak{t}}}_r = \frac{\mathfrak{t}_l^*}{\mathfrak{D}^*}. \tag{142}$$

## 8 $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-Symmetric Scattering Systems

A physical system that involves the scattering of a scalar wave in one dimension is said to be *$\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetric* if its reflection and transmission amplitudes are respectively invariant under space reflection, time-reversal, and the combined action of space reflection and time-reversal transformation, i.e.,

$$\mathcal{P}\text{-symmetry} := \; \tilde{\mathfrak{r}}_{l/r} = \mathfrak{r}_{l/r} \;\text{ and }\; \tilde{\mathfrak{t}}_{l/r} = \mathfrak{t}_{l/r}, \tag{143}$$

$$\mathcal{T}\text{-symmetry} := \; \bar{\mathfrak{r}}_{l/r} = \mathfrak{r}_{l/r} \;\text{ and }\; \bar{\mathfrak{t}}_{l/r} = \mathfrak{t}_{l/r}, \tag{144}$$

$$\mathcal{PT}\text{-symmetry} := \; \tilde{\tilde{\mathfrak{r}}}_{l/r} = \mathfrak{r}_{l/r} \;\text{ and }\; \tilde{\tilde{\mathfrak{t}}}_{l/r} = \mathfrak{t}_{l/r}. \tag{145}$$

We can alternatively state the definition of these symmetries in terms of the invariance of the transfer matrix $\mathbf{M}$ or the scattering matrix $\mathbf{S}$ of the system under the action of $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$. In this section we explore the consequences of these symmetries.

According to (132), the $\mathcal{P}$-symmetry of a scattering system implies

$$\mathfrak{r}_l = \mathfrak{r}_r, \qquad \mathfrak{t}_l = \mathfrak{t}_r. \tag{146}$$

Substituting the latter equation in (21), we find $\det \mathbf{M} = 1$. Let us also mention that in view of (49) and (146), the eigenvalues of the $\mathbf{S}$-matrix for $\mathcal{P}$-symmetric systems take the simple form: $\mathfrak{s}_\pm = \mathfrak{t} \pm \mathfrak{r}$ where $\mathfrak{t} := \mathfrak{t}_l = \mathfrak{t}_r$ and $\mathfrak{r} := \mathfrak{r}_l = \mathfrak{r}_r$.

Another obvious consequence of (146) is that $\mathcal{P}$-symmetric systems cannot support unidirectional reflection or unidirectional invisibility.

The delta-function potential (86) provides a simple example of a $\mathcal{P}$-symmetric potential that may not be time-reversal-invariant. As demonstrated by (88), it complies with (146).

We can similarly derive the consequences of $\mathcal{P}_a$-symmetry. This symmetry also implies transmission reciprocity and $\det \mathbf{M} = 1$, but breaks the reciprocity in reflection amplitudes as it yields the following generalization of the first relation in (146).

$$e^{-2iak}\mathfrak{r}_l(k) = e^{2aik}\mathfrak{r}_r(k). \tag{147}$$

Notice however that reciprocity in reflection coefficients, $|\mathfrak{r}_l|^2 = |\mathfrak{r}_r|^2$, persists. A simple example of $\mathcal{P}_a$-symmetric scattering system is that of the barrier potential (56) with $L = 2a$. Clearly in this case the expressions (60) and (61) for the reflection amplitudes agree with (147).

The consequences of the $\mathcal{T}$-symmetry are more interesting. Imposing (144), we can use (133) to deduce

$$\mathfrak{r}_r^* = -\mathfrak{D}^*\mathfrak{r}_l, \qquad \mathfrak{r}_l^* = -\mathfrak{D}^*\mathfrak{r}_r, \qquad \mathfrak{t}_{l/r}^* = \mathfrak{D}^*\mathfrak{t}_{l/r}. \tag{148}$$

The first two of these relations indicate that either both $\mathfrak{r}_{l/r}$ vanish or $|\mathfrak{D}| = 1$. This means that there is some real number $\sigma \in \mathbb{R}$ such that $\mathfrak{D} = e^{i\sigma}$. Substituting this in (148), we can show that

$$\mathfrak{r}_r = -e^{i\sigma}\mathfrak{r}_l^*, \qquad \mathfrak{t}_{l/r} = \epsilon_{l/r}|\mathfrak{t}_{l/r}|e^{i\sigma/2}, \tag{149}$$

where $\epsilon_{l/r}$ are some unspecified signs; $\epsilon_{l/r} \in \{-1, 1\}$. In particular,

$$|\mathfrak{r}_l| = |\mathfrak{r}_r|. \tag{150}$$

This equation proves the following result.

**Theorem 11** *Time-reversal-invariant systems in one dimension cannot support unidirectional reflection or unidirectional invisibility.*

If we insert (149) in the definition of $\mathfrak{D}$, namely (75), and impose $\mathfrak{D} = e^{i\sigma}$, we find

$$|\mathfrak{r}_l|^2 + \epsilon_l\epsilon_r|\mathfrak{t}_l\mathfrak{t}_r| = 1. \tag{151}$$

The following theorem summarizes the content of Eqs. (150) and (151).

**Theorem 12** *The reflection and transmission amplitudes of a time-reversal-invariant scattering system in one-dimension satisfy*

$$|\mathfrak{r}_l(k)|^2 = |\mathfrak{r}_r(k)|^2 = 1 \pm |\mathfrak{t}_l(k)\mathfrak{t}_r(k)|, \tag{152}$$

*where $k \in \mathbb{R}^+$ and the unspecified sign on the right-hand side is to be taken negative whenever the system has reciprocal transmission, i.e., $\mathfrak{t}_l(k) = \mathfrak{t}_r(k)$.*

If $\mathfrak{t}_r = \mathfrak{t}_l$, which is for example the case for systems that are both $\mathcal{T}$- and $\mathcal{P}$-symmetric or described by a real scattering potential, $\epsilon_l = \epsilon_r$ and we can write (151) as

$$|\mathfrak{r}_{l/r}|^2 + |\mathfrak{t}|^2 = 1, \tag{153}$$

where again $\mathfrak{t} := \mathfrak{t}_l = \mathfrak{t}_r$. Equation (153) is usually derived for real scattering potentials using the unitarity of the time-evolution generated by the corresponding standard Hamiltonian (114). It is therefore often called the *unitarity relation*. The derivation we have offered here is more general, for it relies on the transmission reciprocity and time-reversal-invariance. Removing the first of these conditions, we arrive at (152) which is a mild generalization of the unitarity relation (153). Equation (152) apply, for example, to the scattering problem defined by the time-independent Schrödinger equation for the Hamiltonian operator:

$$H = (I + e^{-\mu\hat{x}^2})\left[\frac{\hat{p}^2}{2m} + v(\hat{x})\right],$$

where $\mu$ is a positive real parameter, and $v(x)$ is a real and even scattering potential. Note that this Hamiltonian is both $\mathcal{P}$- and $\mathcal{T}$-symmetric but not Hermitian.[17]

The unitarity relation (153), which holds for time-reversal-invariant systems with reciprocal transmission, in general, and real scattering potentials in particular, implies that the reflection and transmission coefficients of the system cannot exceed 1; $|\mathfrak{r}(k)|^2 \leq 1$ and $|\mathfrak{t}(k)|^2 \leq 1$ for all $k \in \mathbb{R}^+$. This means that *these systems do not amplify the transmitted or reflected waves*. In particular, we have:

**Theorem 13** *If a time-reversal-invariant scattering system in one dimension has reciprocal transmission, it cannot have spectral singularities.*

It is for this reason that spectral singularities do not appear in the study of unitary quantum systems described by standard Hamiltonian operators.

Time-reversal-invariant systems violating reciprocity in transmission can have spectral singularities. A simple example is a single-center point interaction (7) with $n = 1$, $c_1 = 0$, and

[17] The scattering problem for this Hamiltonian operator is equivalent to that of the energy-dependent scattering potential $v(x,k) := 2mv(x) + k^2/(1 + e^{\mu x^2})$. This is because we can write $H\psi(x) = E\psi(x)$ in the form $-\psi''(x) + v(x,k)\psi(x) = k^2\psi(x)$ where $k := \sqrt{E}$.

$$\mathbf{B} = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R}, \quad \beta\gamma > 0. \tag{154}$$

It is easy to see that the system described by this point interaction is time-reversal-invariant, because $\mathbf{B}$ is a real matrix [41]. Furthermore, we can compute its transfer matrix using (27) and find out that for this system $M_{22}(k) = \beta k^2 - \gamma$. Therefore, it has a spectral singular $k_0^2 = \gamma/\beta$. Note also that because $\det \mathbf{M} = \det \mathbf{B} = -\alpha^2 - \beta\gamma < 0$, $\det \mathbf{M} \neq 1$ which shows that it has nonreciprocal transmission.

We can also characterize time-reversal symmetry in terms of the restrictions it imposes on the transfer and scattering matrices. These have the following simple form.

$$\mathbf{M}^* = \boldsymbol{\sigma}_1 \mathbf{M} \boldsymbol{\sigma}_1, \qquad \mathbf{S}^* = \boldsymbol{\sigma}_1 \mathbf{S}^{-1} \boldsymbol{\sigma}_1. \tag{155}$$

Because $\det \boldsymbol{\sigma}_1 = -1$, the first of these equations implies that $\det \mathbf{M}$ must be real while the second reproduces the result that $\det \mathbf{S}$ is unimodular; $|\det \mathbf{S}| = 1$.

Let us examine the eigenvalues of $\mathbf{S}$ for time-reversal-invariant systems. In view of (49), (149), and (151), these are given by

$$\mathfrak{s}_+ = (\tau + \sqrt{\tau^2 - 1})e^{i\sigma/2}, \quad \mathfrak{s}_- = (\tau - \sqrt{\tau^2 - 1})e^{i\sigma/2} = \frac{e^{i\sigma/2}}{\tau + \sqrt{\tau^2 - 1}}, \tag{156}$$

where

$$\tau(k) := \frac{\epsilon_l |\mathfrak{t}_l(k)| + \epsilon_r |\mathfrak{t}_r(k)|}{2}. \tag{157}$$

It is not difficult to see that $|\mathfrak{s}_\pm| = 1$ if and only if

$$|\tau| \leq 1. \tag{158}$$

If $|\tau(k)| \leq 1$ for all $k \in \mathbb{R}^+$, we say that the time-reversal symmetry of the system is *exact* or *unbroken*. If $|\tau| > 1$ for some $k \in \mathbb{R}^+$, we say that the system has a *broken time-reversal symmetry*.

To examine the physical meaning of exact time-reversal symmetry, we examine the consequences of (158). First we use (157) to write it in the form

$$|\mathfrak{t}_l|^2 + |\mathfrak{t}_r|^2 + 2\epsilon_l\epsilon_r |\mathfrak{t}_l \mathfrak{t}_r| \leq 4. \tag{159}$$

With the help of (151), we can express this equation as

$$\frac{|\mathfrak{t}_l|^2 + |\mathfrak{t}_r|^2}{2} \leq 1 + |\mathfrak{r}_l|^2. \tag{160}$$

If $\epsilon_l\epsilon_r = 1$, (151), (150), and (160) imply

$$|\mathfrak{r}_{l/r}|^2 \leq 1, \qquad\qquad |\mathfrak{t}_l|^2 + |\mathfrak{t}_r|^2 \leq 4. \tag{161}$$

Therefore similarly to the unitary systems the reflection and transmission amplitudes are bounded functions, and the system cannot involve spectral singularities.

If a system has a broken time-reversal symmetry, there is some $k \in \mathbb{R}^+$ such that $|\tau(k)| > 1$. In this case, (151) implies

$$\frac{|\mathfrak{t}_l(k)|^2 + |\mathfrak{t}_r(k)|^2}{2}| > 1 + |\mathfrak{r}_l|^2 \geq 1. \tag{162}$$

Furthermore because $\sqrt{\tau^2 - 1}$ is real and nonzero, (156) implies $|\mathfrak{s}_\pm| \neq 1$ and $\mathfrak{s}_- = 1/\mathfrak{s}_+^*$.

Equation (151) which reveals various properties of the time-reversal-invariant scattering systems has a rather interesting equivalent that does not involve the unspecified signs $\epsilon_{l/r}$. To derive this, first we use (74) and the fact that $\mathfrak{D}(k) = e^{i\sigma(k)}$ to show that

$$\mathfrak{r}_{l/r}(-k) = -e^{-i\sigma(k)}\mathfrak{r}_{r/l}(k), \qquad\qquad \mathfrak{t}_{l/r}(-k) = e^{-i\sigma(k)}\mathfrak{t}_{l/r}(k). \tag{163}$$

These relations have the following straightforward implications:

$$|\mathfrak{r}_{l/r}(-k)| = |\mathfrak{r}_{r/l}(k)|, \qquad\qquad |\mathfrak{t}_{l/r}(-k)| = |\mathfrak{t}_{l/r}(k)|, \tag{164}$$

$$\mathfrak{r}_{l/r}(-k)\mathfrak{r}_{l/r} + \mathfrak{t}_{l/r}(-k)\mathfrak{t}_{r/l}(k) = 1, \tag{165}$$

where we have made use of the definition of $\mathfrak{D}(k)$, i.e., (75), and the fact that $\mathfrak{D}(k) = e^{i\sigma(k)}$.

It is important to notice that our derivation of Equations (163), (164), and (165) only uses the fact that $|\mathfrak{D}(k)| = 1$, which is much less restrictive than the time-reversal symmetry of the system. We state this result as a theorem:

**Theorem 14** *Equations (164) and (165) hold for any scattering system whose reflection and transmission amplitudes satisfy $|\mathfrak{t}_l(k)\mathfrak{t}_r(k) - \mathfrak{r}_l(k)\mathfrak{r}_r(k)| = 1$, i.e., $|\mathfrak{D}(k)| = 1$.*[18]

Next, we examine the implications of $\mathcal{PT}$-symmetry. In view of (142) and (145), the reflection and transmission amplitudes of $\mathcal{PT}$-symmetric scattering systems satisfy

$$\mathfrak{r}_{l/r}^* = -\mathfrak{D}^*\mathfrak{r}_{l/r}, \qquad\qquad \mathfrak{t}_{l/r}^* = \mathfrak{D}^*\mathfrak{t}_{l/r}. \tag{166}$$

[18] An extension of this theorem to more general scattering systems is given in [52].

If we complex-conjugate both sides of (75) and use (166) in the right-hand side of the resulting equation, we find $\mathfrak{D}^* = \mathfrak{D}^{*2}\mathfrak{D}$, which means $|\mathfrak{D}| = 1$. In view of Theorem 14, this shows that, similarly to time-reversal-invariant systems, $\mathcal{PT}$-symmetric systems satisfy the identities (164) and (165).[19]

Because $|\mathfrak{D}| = 1$, $\mathfrak{D} = e^{i\sigma}$ for some $\sigma \in \mathbb{R}$. Using this relation in (166), we can show that

$$\mathfrak{r}_{l/r} = i\eta_{l/r}e^{i\sigma/2}|\mathfrak{r}_{l/r}|, \qquad \mathfrak{t}_{l/r} = \epsilon_{l/r}e^{i\sigma/2}|\mathfrak{t}_{l/r}|, \tag{167}$$

where $\eta_{l/r}, \epsilon_{l/r} \in \{-1, 1\}$. Now, we substitute these relations in (75) and make use of $\mathfrak{D} = e^{i\sigma}$ to conclude that

$$\epsilon_l\epsilon_r|\mathfrak{t}_l\mathfrak{t}_r| + \eta_l\eta_r|\mathfrak{r}_l\mathfrak{r}_r| = 1. \tag{168}$$

According to this equation, $\epsilon_l\epsilon_r = -1$ implies $\eta_l\eta_r = 1$ and $\eta_l\eta_r = -1$ implies $\epsilon_l\epsilon_r = 1$. These observations prove the following theorem.

**Theorem 15** *For all $k \in \mathbb{R}^+$, the reflection and transmission amplitudes of a $\mathcal{PT}$-symmetric scattering system in one-dimension satisfy either*

$$|\mathfrak{t}_l(k)\mathfrak{t}_r(k)| = -1 + |\mathfrak{r}_l(k)\mathfrak{r}_r(k)|, \tag{169}$$

*or*

$$|\mathfrak{t}_l(k)\mathfrak{t}_r(k)| = 1 \pm |\mathfrak{r}_l(k)\mathfrak{r}_r(k)|. \tag{170}$$

*If the system has reciprocal transmission, i.e., $\mathfrak{t}_l(k) = \mathfrak{t}_r(k)$, only the second of these relations holds. In this case, we have*

$$|\mathfrak{t}(k)|^2 \pm |\mathfrak{r}_l(k)\mathfrak{r}_r(k)| = 1. \tag{171}$$

*If the system has reciprocal reflection, i.e., $\mathfrak{r}_l(k) = \mathfrak{r}_r(k)$, (169) is not excluded but the unspecified sign on the right-hand side of (170) is to be taken negative, i.e., it reads*

$$|\mathfrak{t}_l(k)\mathfrak{t}_r(k)| + |\mathfrak{r}(k)|^2 = 1. \tag{172}$$

[19] Equations (164) was originally conjectures in [1] for $\mathcal{PT}$-symmetric scattering potentials based on evidence provided by the study of a complexified Scarf II potential. It was subsequently proven in [48] for general $\mathcal{PT}$-symmetric scattering potentials which respect transmission reciprocity.

For a scattering system defined by a $\mathcal{PT}$-symmetric scattering potential, Theorem 2 ensures the reciprocity in transmission. Therefore, $\mathcal{PT}$-symmetric scattering potentials satisfy (171), [11].

Next, we examine the effect of $\mathcal{PT}$-symmetry on the transfer and scattering matrices. It is easy to show that for $\mathcal{PT}$-symmetric systems,

$$\mathbf{M}^* = \mathbf{M}^{-1}, \qquad\qquad \mathbf{S}^\dagger = \boldsymbol{\sigma}_1\mathbf{S}^{-1}\boldsymbol{\sigma}_1, \tag{173}$$

where $\mathbf{S}^\dagger$ is the conjugate-transpose or Hermitian-conjugate of $\mathbf{S}$. The first of these relations follows from (138) and implies that $\det\mathbf{M}$ is unimodular;

$$|\det\mathbf{M}| = 1. \tag{174}$$

The second is a consequence of (48) and (166). Because $\boldsymbol{\sigma}_1^{-1} = \boldsymbol{\sigma}_1$, we can write it in the form $\mathbf{S}^\dagger = \boldsymbol{\sigma}_1\mathbf{S}^{-1}\boldsymbol{\sigma}_1^{-1}$. This indicates that $\mathbf{S}$ is a $\boldsymbol{\sigma}_1$*-pseudo-unitary matrix* [36], i.e., if we identify the elements of $\mathbb{C}^2$ with $2\times 1$ matrices and view $\boldsymbol{\sigma}_1$ and $\mathbf{S}$ as linear operators acting on them, then $\mathbf{S}$ preserves the indefinite inner product:

$$\langle\mathbf{a},\mathbf{b}\rangle_{\sigma_1} := \langle\mathbf{a}|\boldsymbol{\sigma}_1\mathbf{b}\rangle = \mathbf{a}^\dagger\boldsymbol{\sigma}_1\mathbf{b} = a_1^*b_2 + a_2^*b_1,$$

where $\mathbf{a} = [a_1\ a_2]^T$ and $\mathbf{b} = [b_1\ b_2]^T$ are arbitrary $2\times 1$ complex matrices, and a superscript "T" on a matrix labels its transpose.[20] Because the $\mathbf{S}$-matrix of every $\mathcal{PT}$-symmetric scattering potential is $\boldsymbol{\sigma}_1$-pseudo-unitary, Eq. (171) is sometimes called the *pseudo-unitarity relation*.

In general, an invertible square matrix $\boldsymbol{\mathcal{U}}$ is said to be *pseudo-unitary*, if there is an invertible Hermitian matrix $\boldsymbol{\eta}$ such that $\boldsymbol{\mathcal{U}}^\dagger = \boldsymbol{\eta}\boldsymbol{\mathcal{U}}^{-1}\boldsymbol{\eta}^{-1}$. Pseudo-unitary matrices have the property that the inverse of the complex-conjugate of their eigenvalues are also eigenvalues, i.e., if $\mathfrak{s}$ is an eigenvalue of a pseudo-unitary matrix, either $|\mathfrak{s}| = 1$ or $1/\mathfrak{s}^*$ is also an eigenvalue [36]. As we show above this condition applies also for the eigenvalues of the $\mathbf{S}$-matrix for time-reversal-invariant systems. We can check its validity for the $\mathbf{S}$-matrix of $\mathcal{PT}$-symmetric systems by a direct calculation of its eigenvalues. Inserting (167) in (49) and making use of (168), we find that the expression for $\mathfrak{s}_\pm$ coincides with the one we obtain for the time-reversal-invariant systems, namely (156). Therefore, again either $|\tau| \leq 1$ in which case $|\mathfrak{s}_\pm| = 1$, or $|\tau| > 1$ in which case $|\mathfrak{s}_\pm| \neq 1$ and $\mathfrak{s}_- = 1/\mathfrak{s}_+^*$.

Following the terminology we employed in our discussion of time-reversal symmetry, we use the sign of $1-|\tau|$ to introduce the notions of exact and broken $\mathcal{PT}$-symmetry. If for all $k\in\mathbb{R}^+$, $1-\tau(k)| \geq 0$ so that $|\mathfrak{s}_\pm(k)| = 1$, we say that the system has an *exact or unbroken $\mathcal{PT}$-symmetry*. If this is not the case we say that its $\mathcal{PT}$-symmetry is *broken*. This terminology should not be confused with the

[20] For a $2\times 2$ matrix $\mathbf{A}$, the condition of being $\boldsymbol{\sigma}_1$-pseudo-unitary is equivalent to the requirement that $e^{i\pi\boldsymbol{\sigma}_2/4}\mathbf{A}e^{-i\pi\boldsymbol{\sigma}_2/4}$ belong to the pseudo-unitary group $U(1,1)$, where $\boldsymbol{\sigma}_2$ is the second Pauli matrix.

one employed in the study of $\mathcal{PT}$-symmetric Hamiltonian operators $H$ that have a discrete spectrum. For these systems unbroken $\mathcal{PT}$-symmetry means the existence of a complete set of eigenvectors of $H$ that are also eigenvectors of $\mathcal{PT}$. This in turn implies the reality of the spectrum of $H$, [2]. Scattering theory for a $\mathcal{PT}$-symmetric Hamiltonian is sensible only if its spectrum contains a real continuous part that covers the positive real axis in the complex plane. In particular it may or may not have nonreal eigenvalues.[21]

If for some $k \in \mathbb{R}^+$, a $\mathcal{PT}$-symmetric system has reciprocal transmission, $\tau(k) = |\mathfrak{t}(k)|$. Therefore the condition $|\tau| \leq 1$ puts an upper bound of 1 on the transmission coefficient $|\mathfrak{t}(k)|^2$. This in turn implies that the unspecified sign in (171) must be taken positive and $|\mathfrak{r}_l(k)\mathfrak{r}_r(k)| \leq 1$. As a result, the system cannot amplify reflected or transmitted waves having wavenumber $k$. In particular $k^2$ cannot be a spectral singularity. In summary, *for a system with reciprocal transmission, such as those described by a scattering potential, exactness of $\mathcal{PT}$-symmetry forbids amplification of the reflected and transmitted waves and spectral singularities.*

An important advantage of $\mathcal{PT}$-symmetry over $\mathcal{P}$- and $\mathcal{T}$-symmetries, is that it does not imply the equality of the left and right reflection amplitudes. Therefore *unidirectional reflection and unidirectional invisibility are not forbidden by $\mathcal{PT}$-symmetry*. In fact, it turns out that it is easier to achieve unidirectional reflectionlessness and invisibility in the presence of $\mathcal{PT}$-symmetry than in its absence. This has to do with the following result that is a straightforward consequence of (142).

**Theorem 16** *The equations characterizing unidirectional invisibility, namely*

$$\mathfrak{r}_{l/r}(k) = 0 \neq \mathfrak{r}_{r/l}, \qquad\qquad \mathfrak{t}_{l/r}(k) = 1, \tag{175}$$

*are invariant under the $\mathcal{PT}$-transformation.*

For a $\mathcal{PT}$-symmetric system the equations of unidirectional invisibility enjoy the same symmetry as that of the underlying wave equation. This leads to enormous practical simplifications in constructing specific unidirectionally invisible models. It does not however imply that $\mathcal{PT}$-symmetry is a necessary condition for unidirectional reflection or invisibility [43].

## 9 Time-Reversed and Self-Dual Spectral Singularities

Consider a linear scattering system $\mathcal{S}$ with an invertible transfer matrix $\mathbf{M}(k)$. Then spectral singularities of this system are determined by the real and positive zeros of $M_{22}(k)$. According to (130), $M_{11}(k) = 0$ if and only if $\overline{M}_{22}(k) = 0$. This in

[21] We use the term "eigenvalue" to mean an element of the point spectrum of $H$ which has a square-integrable eigenfunction.

turn means that the real and positive zeros of $M_{11}(k)$ give the spectral singularities of the time-reversed system $\overline{\mathcal{S}}$. We will refer to these as the *time-reversed spectral singularities* of $\mathcal{S}$.

At a time-reversed spectral singularity the Jost solutions of the time-reversed system become linearly dependent and satisfy purely outgoing boundary conditions at $x = \pm\infty$. This suggests the presence of solutions of the wave equation for the system $\mathcal{S}$ that satisfy purely incoming asymptotic boundary conditions. To see this, first we note that according to Eq. (14) whenever $M_{11}(k) = 0$, we can have a solution $\psi(x)$ of the wave equation satisfying (1) with $A_+(k) = B_-(k) = 0$, i.e.,

$$\psi(x) \to N_\pm(k)e^{\mp ikx} \qquad \text{for} \qquad x \to \pm\infty, \tag{176}$$

where $N_\pm(k)$ are nonzero complex coefficients satisfying

$$N_+(k) = M_{21}(k)N_-(k). \tag{177}$$

In other words, $\psi(x)$ satisfies the asymptotic boundary conditions (1) with

$$A_-(k) = N_-(k), \qquad B_-(k) = 0, \qquad A_+(k) = 0, \qquad B_+(k) = N_+(k).$$

If we substitute these in the first equation in (42) and recall that $\mathbf{S}_1 = \mathbf{S}$, we find that

$$\mathbf{S}(k)\begin{bmatrix} N_-(k) \\ N_+(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This shows that $[N_-(k)\ N_+(k)]^T$ is an eigenvector of $\mathbf{S}(k)$ with eigenvalue zero. In particular, one of the eigenvalues of $\mathbf{S}(k)$ vanishes.

The existence of a solution of the wave equation having the asymptotic expression (176) means that the scatterer will absorb any pair of incident left- and right-going waves whose complex amplitude $N_\pm(k)$ are related by (177). This phenomenon is called *coherent perfect absorption* [6, 27, 75]. In the study of effectively one-dimensional optical systems, spectral singularities correspond to the initiation of laser oscillations in a medium with gain, i.e., a laser, while their time-reversal give rise to perfect absorption of finely tuned coherent incident beams by a medium with loss. The latter is sometimes called an *antilaser*.

It may happen that a particular wavenumber $k_0$ is a common zero of both $M_{11}(k)$ and $M_{22}(k)$. In this case, we call $k_0^2$ a *self-dual spectral singularity* [42]. At a self-dual spectral singularity the wave equation admits both purely outgoing and purely incoming solutions. This means that if the system is not subject to any incident wave, it will amplify the background noise and begin emitting outgoing waves of wavenumber $k_0$. But if it is subject to a pair of left- and right-going incident waves with wavenumber $k_0$ and complex amplitudes satisfying (177) for $k = k_0$, then it will absorb them completely. In its optical realizations this corresponds to a special laser that becomes a coherent perfect absorber (CPA) once it is subject to an appropriate pair of incoming waves. Such a device is called a *CPA-laser*.

For a time-reversal-invariant system we have $M_{11}(k) = M_{22}(k)^*$. Therefore every spectral singularity is self-dual. But according to Theorem 13 spectral singularities are forbidden for time-reversal-invariant systems with reciprocal transmission. This excludes real scattering potentials. There are however nonreal potentials that admit self-dual spectral singularities. Principal examples are $\mathcal{PT}$-symmetric scattering potentials [7, 28, 77]. According to (140), for every $\mathcal{PT}$-symmetric scattering system,

$$M_{11}(k) = \det \mathbf{M}(k) M_{22}(k)^*.$$

This proves the following theorem.

**Theorem 17** *Spectral singularities of every $\mathcal{PT}$-symmetric scattering system are self-dual.*

This does not however exclude the possibility of having non-$\mathcal{PT}$-symmetric systems with self-dual spectral singularities. Simple examples of the latter are examined in [19, 22, 42].

## 10 Summary and Concluding Remarks

Scattering of waves can be studied using a general framework where the asymptotic solutions of the relevant wave equation are plane waves. This point of view is analogous to the general philosophy leading to the **S**-matrix formulation of scattering in the late 1930s. In one dimension, the transfer matrix proves to be a much more powerful tool than the **S**-matrix. We have therefore offered a detailed discussion of the transfer matrix and used it to introduce and explore the implications of $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetry. This is actually quite remarkable, for we could derive a number of interesting and useful quantitative results regarding the consequences of such symmetries without actually imposing them on the wave equation. These results apply to scattering phenomena modeled using local as well as nonlocal potentials and point interactions. The general setup we offer in Sect. 1 can also be used in the study of the scattering of a large class of nonlinear waves that are asymptotically linear. The results we derived using the transfer matrix may not however extend to such waves.

The recent surge of interest in the properties of $\mathcal{PT}$-symmetric scattering potentials has led to the study of remarkable effects such as unidirectional invisibility, optical spectral singularities, and coherent perfect absorption. The global approach to scattering that we have outlined here allows for a precise description of these concepts for a general class of scattering systems that cannot be described using a local scattering potential. In particular, we have derived specific conditions imposed by $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetry on the presence of nonreciprocal transmission and reflection, spectral singularities and their time-reversal, and unidirectional reflectionlessness and invisibility.

A recent development that we have not covered in the present text is the construction of a transfer matrix for potential scattering in two and three dimensions [31]. This has led to the discovery of a large class of exactly solvable multidimensional scattering potentials [33], and allowed for the extension of the notions of spectral singularity and unidirectional invisibility to higher dimensions [31, 32]. A particularly remarkable application of the multidimensional transfer matrix is the construction of scattering potentials in two dimensions that display perfect broadband invisibility below a tunable critical frequency [34].

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