CHAPTER 11

DYNAMICS OF RIGID BODIES

11.1 INTRODUCTION

We define a rigid body as a collection of particles whose relative distances are constrained to remain absolutely fixed. Such bodies do not exist in nature, because the ultimate component particles composing every body (the atoms) are always undergoing some relative motion. This motion, however, is microscopic, and it therefore usually may be ignored when describing the macroscopic motion of the body. However, macroscopic displacement within the body (such as elastic deformations) can take place. For many bodies of interest, we can safely neglect the changes in size and shape caused by such deformations and obtain equations of motion valid to a high degree of accuracy.

It should also be clear that there is a relativistic limitation to the concept of an absolutely rigid body. Consider, for example, a long bar of some material. If we strike a blow at one end of the bar and if the bar were absolutely rigid, the effect would be felt instantaneously at the opposite end. But this corresponds to the transmission of a signal with an infinite velocity—a situation that, from relativity theory, we know is impossible. (Actually, the velocity of transmission of such a signal in a metal bar is rather low compared with the velocity of light—$\sim 10^7$ m/s—and depends on the elastic properties of the material.)

We here use the idealized concept of a rigid body as a collection of discrete particles or as a continuous distribution of matter interchangeably. The only change is the replacement of summations over particles by integrations over mass density distributions. The equations of motion are equally valid for either viewpoint.

To describe the motion of a rigid body, we use two coordinate systems—an inertial frame and a coordinate system fixed with respect to the body. Six quantities must be specified to denote the position of the body. These can be taken to be the coordinates of the center of mass (which can often conveniently be made to coincide with the origin of the body coordinate system) and three independent angles that give the orientation of the body coordinate system with respect to the fixed (or inertial) system.* The three independent angles can conveniently be taken to be the **Eulerian angles**, described in Section 11.7.

It should be intuitively obvious that any arbitrary finite motion of a rigid body can be considered to be the sum of two independent motions—a linear translation of some point of the body plus a rotation about that point.† If the point is chosen to be the center of mass of the body, then such a separation of the motion into two parts allows the use of the development in Chapter 9, which indicates that the angular momentum (see Equation 9.23) and the kinetic energy (see Equation 9.39) can be separated into portions relating to the motion *of* the center of mass and to the motion *around* the center of mass.

If the potential energy can also be separated (as is always the case, for example, for the potential energy in a uniform force field), then the Lagrangian separates, and the entire problem conveniently divides into two parts, one involving only translation and the other only rotation. Each portion of the problem can then be solved independently of the other.‡ This type of separation is essential for a relatively uncomplicated description of rigid-body motion.

11.2 INERTIA TENSOR

We now direct our attention to a rigid body composed of n particles of masses m_α, $\alpha = 1, 2, 3, \ldots, n$. If the body rotates with an instantaneous angular velocity $\boldsymbol{\omega}$ about some point fixed with respect to the body coordinate system and if this point moves with an instantaneous linear velocity $\mathbf{V}$ with respect to the fixed coordinate system, then the instantaneous velocity of the αth particle in the fixed system can be obtained by using Equation 10.17. But we are now considering a rigid body, so

$$\mathbf{v}_r = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} \equiv 0$$

Therefore,

$$\boxed{\mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha} \tag{11.1}$$

* In this chapter, we use the designation *body system* in place of the term *rotating system* used in the preceding chapter. The term *fixed system* will be retained.

† *Chasles' theorem*, which is even more general than this statement (it says that the line of translation and the axis of rotation can be made to coincide), was proven by the French mathematician Michel Chasles (1793–1880) in 1830. The proof is given, e.g., by E. T. Whittaker (Wh37, p. 4).

‡ This important point was first realized by Euler in 1749.

where the subscript f, denoting the fixed coordinate system, has been deleted from the velocity $\mathbf{v}_\alpha$, it now being understood that all velocities are measured in the fixed system. All velocities with respect to the rotating or body system now vanish because the body is *rigid*.

Because the kinetic energy of the αth particle is given by

$$T_\alpha = \tfrac{1}{2} m_\alpha v_\alpha^2 \tag{11.2}$$

we have, for the total kinetic energy,

$$T = \tfrac{1}{2} \sum_\alpha m_\alpha (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 \tag{11.3}$$

Expanding the squared term, we find

$$T = \tfrac{1}{2} \sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{r}_\alpha + \tfrac{1}{2} \sum_\alpha m_\alpha (\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 \tag{11.4}$$

This is a general expression for the kinetic energy and is valid for any choice of the origin from which the vectors $\mathbf{r}_\alpha$ are measured. But if we make the origin of the body coordinate system coincide with the center of mass of the object, a considerable simplification results. First, we note that in the second term on the right-hand side of this equation neither $\mathbf{V}$ nor $\boldsymbol{\omega}$ is characteristic of the αth particle, and therefore, these quantities may be taken outside the summation:

$$\sum_\alpha m_\alpha \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{r}_\alpha = \mathbf{V} \cdot \boldsymbol{\omega} \times \left(\sum_\alpha m_\alpha \mathbf{r}_\alpha \right) \tag{11.5}$$

But now the term

$$\sum_\alpha m_\alpha \mathbf{r}_\alpha = M\mathbf{R}$$

is the center-of-mass vector (see Equation 9.3), which vanishes in the body system because the vectors $\mathbf{r}_\alpha$ are measured from the center of mass. The kinetic energy can then be written as

$$T = T_{\text{trans}} + T_{\text{rot}}$$

where

$$\boxed{T_{\text{trans}} = \tfrac{1}{2} \sum_\alpha m_\alpha V^2 = \tfrac{1}{2} M V^2} \tag{11.6a}$$

$$\boxed{T_{\text{rot}} = \tfrac{1}{2} \sum_\alpha m_\alpha (\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2} \tag{11.6b}$$

T_{trans} and T_{rot} designate the translational and rotational kinetic energies, respectively. Thus, the kinetic energy separates into two independent parts.

The rotational kinetic energy term can be evaluated by noting that

$$(\mathbf{A} \times \mathbf{B})^2 = (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$$
$$= A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2$$

Therefore,

$$T_{\text{rot}} = \tfrac{1}{2} \sum_\alpha m_\alpha [\omega^2 r_\alpha^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)^2] \tag{11.7}$$

We now express T_{rot} by using the components ω_i and $r_{\alpha,i}$ of the vectors $\boldsymbol{\omega}$ and $\mathbf{r}_\alpha$. We also note that $\mathbf{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ in the body system, so we can write $r_{\alpha,i} = x_{\alpha,i}$. Thus,

$$T_{\text{rot}} = \tfrac{1}{2} \sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2 \right) \left(\sum_k x_{\alpha,k}^2 \right) - \left(\sum_i \omega_i x_{\alpha,i} \right) \left(\sum_j \omega_j x_{\alpha,j} \right) \right] \tag{11.8}$$

Now, we can write $\omega_i = \sum_j \omega_j \delta_{ij}$, so that

$$T_{\text{rot}} = \tfrac{1}{2} \sum_\alpha \sum_{i,j} m_\alpha \left[\omega_i \omega_j \delta_{ij} \left(\sum_k x_{\alpha,k}^2 \right) - \omega_i \omega_j x_{\alpha,i} x_{\alpha,j} \right]$$
$$= \tfrac{1}{2} \sum_{i,j} \omega_i \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \tag{11.9}$$

If we define the ijth element of the sum over α to be I_{ij},

$$\boxed{I_{ij} \equiv \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)} \tag{11.10}$$

then we have

$$\boxed{T_{\text{rot}} = \tfrac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j} \tag{11.11}$$

This equation in its most restricted form becomes

$$T_{\text{rot}} = \tfrac{1}{2} I \omega^2 \tag{11.12}$$

where I is the (scalar) moment of inertia about the axis of rotation. This equation will be recognized as the familiar expression for the rotational kinetic energy given in elementary treatments.

The nine terms I_{ij} constitute the elements of a quantity we designated by $\{\mathbf{I}\}$. In form, $\{\mathbf{I}\}$ is similar to a 3×3 matrix. It is the proportionality factor between the rotational kinetic energy and the angular velocity and has the dimensions (mass) $\times$ (length)^2. Because $\{\mathbf{I}\}$ relates two quite different physical quantities, we expect that it is a member of a somewhat higher class of functions than has heretofore been encountered. Indeed, $\{\mathbf{I}\}$ is a tensor and is known as the **inertia tensor**.* Note, however, that T_{rot} can be calculated without regard to any of the special properties of tensors, by using Equation 11.9, which completely specifies the necessary operations.

The elements of $\{\mathbf{I}\}$ can be obtained directly from Equation 11.10. We write the elements in a 3 × 3 array for clarity:

$$\{\mathbf{I}\} = \begin{Bmatrix} \sum_\alpha m_\alpha(x_{\alpha,2}^2 + x_{\alpha,3}^2) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_\alpha m_\alpha(x_{\alpha,1}^2 + x_{\alpha,3}^2) & -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha(x_{\alpha,1}^2 + x_{\alpha,2}^2) \end{Bmatrix} \tag{11.13a}$$

Equation 11.10 is a compact way to write the inertia tensor components, but Equation 11.13a is an imposing equation. By using components $(x_\alpha, y_\alpha, z_\alpha)$ instead of $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ and letting $r_\alpha^2 = x_\alpha^2 + y_\alpha^2 + z_\alpha^2$, Equation 11.13a can be written as

$$\{\mathbf{I}\} = \begin{Bmatrix} \sum_\alpha m_\alpha(r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ -\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha(r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha(r_\alpha^2 - z_\alpha^2) \end{Bmatrix} \tag{11.13b}$$

which is less imposing and more recognizable. We continue, however, with the $x_{\alpha,i}$ notation because of its utility.

The diagonal elements, I_{11}, I_{22}, and I_{33}, are called the **moments of inertia** about the x_1-, x_2-, and x_3-axes, respectively, and the negatives of the off-diagonal elements I_{12}, I_{13}, and so forth, are termed the **products of inertia**.† It should be clear that the inertia tensor is symmetric; that is,

$$I_{ij} = I_{ji} \tag{11.14}$$

and, therefore, that there are only six independent elements in $\{\mathbf{I}\}$. Furthermore, the inertia tensor is composed of additive elements; the inertia tensor for a body can be considered to be the sum of the tensors for the various portions of the body. Therefore, if we consider a body as a continuous distribution of matter with mass density $\rho = \rho(\mathbf{r})$, then

$$\boxed{I_{ij} = \int_V \rho(\mathbf{r}) \left(\delta_{ij} \sum_k x_k^2 - x_i x_j \right) dv} \tag{11.15}$$

* The true test of a tensor lies in its behavior under a coordinate transformation (see Section 11.6).

† Introduced by Huygens in 1673; Euler coined the name.

where $dv = dx_1\, dx_2\, dx_3$ is the element of volume at the position defined by the vector $\mathbf{r}$, and where V is the volume of the body.

EXAMPLE 11.1

Calculate the inertia tensor of a homogeneous cube of density ρ, mass M, and side of length b. Let one corner be at the origin, and let three adjacent edges lie along the coordinate axes (Figure 11-1). (For this choice of the coordinate axes, it should be obvious that the origin does not lie at the center of mass; we return to this point later.)

Solution: According to Equation 11.15, we have

$$I_{11} = \rho \int_0^b dx_3 \int_0^b dx_2(x_2^2 + x_3^2) \int_0^b dx_1$$
$$= \tfrac{2}{3}\rho b^5 = \tfrac{2}{3}Mb^2$$
$$I_{12} = -\rho \int_0^b x_1\, dx_1 \int_0^b x_2\, dx_2 \int_0^b dx_3$$
$$= -\tfrac{1}{4}\rho b^5 = -\tfrac{1}{4}Mb^2$$

It should be easy to see that all the diagonal elements are equal and, furthermore, that all the off-diagonal elements are equal. If we define $\beta \equiv Mb^2$, we have

$$\left.\begin{aligned} I_{11} = I_{22} = I_{33} &= \tfrac{2}{3}\beta \\ I_{12} = I_{13} = I_{23} &= -\tfrac{1}{4}\beta \end{aligned}\right\}$$

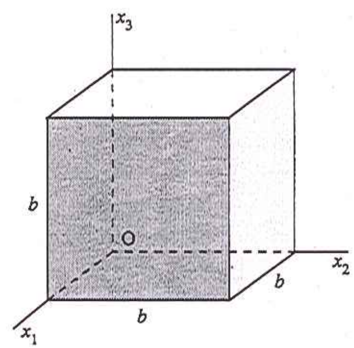


FIGURE 11-1

The moment-of-inertia tensor then becomes

$$\{\mathbf{I}\} = \begin{Bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{Bmatrix}$$

We shall continue the investigation of the moment-of-inertia tensor for the cube in later sections.

11.3 ANGULAR MOMENTUM

With respect to some point O fixed in the body coordinate system, the angular momentum of the body is

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{11.16}$$

The most convenient choice for the position of the point O depends on the particular problem. Only two choices are important: (a) if one or more points of the body are fixed (in the fixed coordinate system), O is chosen to coincide with one such point (as in the case of the rotating top, Section 11.10); (b) if no point of the body is fixed, O is chosen to be the center of mass.

Relative to the body coordinate system, the linear momentum $\mathbf{p}_\alpha$ is

$$\mathbf{p}_\alpha = m_\alpha \mathbf{v}_\alpha = m_\alpha \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

Hence, the angular momentum of the body is

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha) \tag{11.17}$$

The vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$$

can be used to express $\mathbf{L}$:

$$\boxed{\mathbf{L} = \sum_\alpha m_\alpha [r_\alpha^2 \boldsymbol{\omega} - \mathbf{r}_\alpha(\mathbf{r}_\alpha \cdot \boldsymbol{\omega})]} \tag{11.18}$$

The same technique we used to write T_{rot} in tensor form can now be applied here. But the angular momentum is a vector, so for the ith component, we write

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j}\omega_j \right) \\ &= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - \omega_j x_{\alpha,i} x_{\alpha,j} \right) \\ &= \sum_j \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \end{aligned} \tag{11.19}$$

The summation over α can be recognized (see Equation 11.10) as the ijth element of the inertia tensor. Therefore,

$$\boxed{L_i = \sum_j I_{ij}\omega_j} \tag{11.20a}$$

or, in tensor notation,

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} \tag{11.20b}$$

Thus, the inertia tensor relates a *sum* over the components of the angular velocity vector to the ith component of the angular momentum vector. This may at first seem a somewhat unexpected result; for, if we consider a rigid body for which the inertia tensor has nonvanishing off-diagonal elements, then even if $\boldsymbol{\omega}$ is directed along, say, the x_1-direction, $\boldsymbol{\omega} = (\omega_1, 0, 0)$, the angular momentum vector in general has nonvanishing components in all three directions: $\mathbf{L} = (L_1, L_2, L_3)$; that is, the angular momentum vector does not in general have the same direction as the angular velocity vector. (It should be emphasized that this statement depends on $I_{ij} \neq 0$ for $i \neq j$; we return to this point in the next section.)

As an example of $\boldsymbol{\omega}$ and $\mathbf{L}$ not being colinear, consider the rotating dumbbell in Figure 11-2. (We consider the shaft connecting m_1 and m_2 to be weightless and extensionless.) The relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\boldsymbol{\omega}$ is

$$\mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

and the relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\mathbf{L}$ is

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha$$

It should be clear that $\boldsymbol{\omega}$ is directed along the axis of rotation and that $\mathbf{L}$ is perpendicular to the line connecting m_1 and m_2.

We note, for this example, that the angular-momentum vector $\mathbf{L}$ does not remain constant in time but rotates with an angular velocity ω in such a way that it traces out a cone whose axis is the axis of rotation. Therefore $\dot{\mathbf{L}} \neq 0$. But Equation 9.31 states that

$$\dot{\mathbf{L}} = \mathbf{N} \tag{11.21}$$

where $\mathbf{N}$ is the external torque applied to the body. Thus, to keep the dumbbell rotating as in Figure 11-2, we must constantly apply a torque.

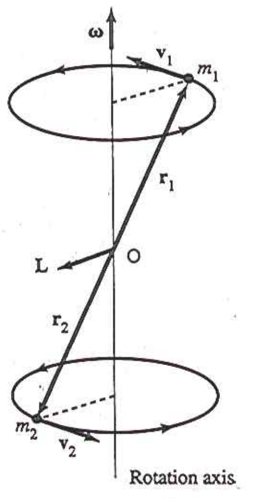


FIGURE 11-2

We can obtain another result from Equation 11.20a by multiplying L_i by $\frac{1}{2}\omega_i$ and summing over i:

$$\frac{1}{2}\sum_i \omega_i L_i = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = T_{rot} \tag{11.22a}$$

where the second equality is just Equation 11.11. Thus,

$$\boxed{T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}} \tag{11.22b}$$

Equations 11.20b and 11.22b illustrate two important properties of tensors. The product of a tensor and a vector yields a vector, as in

$$\mathbf{L} = \{\mathbf{I}\}\cdot\boldsymbol{\omega}$$

and the product of a tensor and two vectors yields a scalar, as in

$$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\cdot\{\mathbf{I}\}\cdot\boldsymbol{\omega}$$

We shall not, however, have occasion to use tensor equations in this form. We use only the summation (or integral) expressions as in Equations 11.11, 11.15, and 11.20a.

EXAMPLE 11.2

Consider the pendulum shown in Figure 11-3 composed of a rigid rod of length b with a mass m_1 at its end. Another mass (m_2) is placed halfway down the rod. Find the frequency of small oscillations if the pendulum swings in a plane.

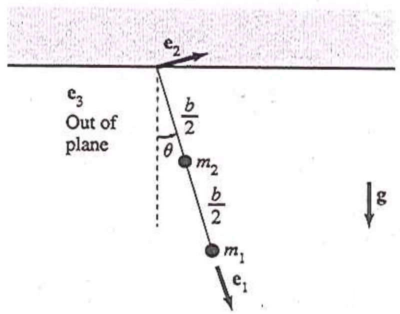


FIGURE 11-3

Solution: We use the methods of this chapter to analyze the system. Let the fixed and body systems have their origin at the pendulum pivot point. Let $\mathbf{e}_1$ be along the rod, $\mathbf{e}_2$ be in the plane, and $\mathbf{e}_3$ be out of the plane (Figure 11-3). The angular velocity is

$$\boldsymbol{\omega} = \omega_3\mathbf{e}_3 = \dot{\theta}\mathbf{e}_3 \tag{11.23}$$

We use Equation 11.10 to find the inertia tensor. All the mass is along $\mathbf{e}_1$, with $x_{1,1} = b$ and $x_{2,1} = b/2$. All other components of $x_{\alpha,k}$ equal zero.

$$I_{ij} = m_1(\delta_{ij}x_{1,1}^2 - x_{1,i}x_{1,j}) + m_2(\delta_{ij}x_{2,1}^2 - x_{2,i}x_{2,j}) \tag{11.24}$$

The inertia tensor, Equation 11.13a, becomes

$$\{\mathbf{I}\} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & m_1b^2 + m_2\dfrac{b^2}{4} & 0 \\ 0 & 0 & m_1b^2 + m_2\dfrac{b^2}{4} \end{Bmatrix} \tag{11.25}$$

We determine the angular momentum from Equation 11.20a:

$$\left.\begin{aligned} L_1 &= 0 \\ L_2 &= 0 \\ L_3 &= I_{33}\omega_3 = \left(m_1b^2 + m_2\frac{b^2}{4}\right)\dot{\theta} \end{aligned}\right\} \tag{11.26}$$

The only external force is gravity, which causes a torque $\mathbf{N}$ on the system. Because $\dot{\mathbf{L}} = \mathbf{N}$, we have

$$\left(m_1b^2 + m_2\frac{b^2}{4}\right)\ddot{\theta}\mathbf{e}_3 = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha \tag{11.27}$$

Because the gravitational force is down,

$$\mathbf{g} = g\cos\theta\mathbf{e}_1 - g\sin\theta\mathbf{e}_2$$

Thus,

$$\mathbf{r}_1 \times \mathbf{F}_1 = b\mathbf{e}_1 \times (\cos\theta\mathbf{e}_1 - \sin\theta\mathbf{e}_2)m_1 g = -m_1 g b\sin\theta\mathbf{e}_3$$

$$\mathbf{r}_2 \times \mathbf{F}_2 = \frac{b}{2}\mathbf{e}_1 \times (\cos\theta\mathbf{e}_1 - \sin\theta\mathbf{e}_2)m_2 g = -m_2 g\frac{b}{2}\sin\theta\mathbf{e}_3$$

Equation 11.27 becomes

$$b^2\left(m_1 + \frac{m_2}{4}\right)\ddot{\theta} = -bg\sin\theta\left(m_1 + \frac{m_2}{2}\right) \tag{11.28}$$

and the frequency of small oscillations is

$$\omega_0^2 = \frac{m_1 + \dfrac{m_2}{2}}{m_1 + \dfrac{m_2}{4}}\frac{g}{b} \tag{11.29}$$

We can check Equation 11.29 by noting that $\omega_0^2 \approx g/b$ for $m_1 \gg m_2$ and $\omega_0^2 \approx 2g/b$ for $m_2 \gg m_1$ as it should.

This example could have just as easily been solved by finding the kinetic energy from Equation 11.22a and using Lagrange's equations of motion. We would then have

$$T_{\text{rot}} = \tfrac{1}{2}\omega_3 L_3 = \tfrac{1}{2}\omega_3^2 I_{33}$$

$$= \frac{1}{2}\left(m_1 b^2 + m_2\frac{b^2}{4}\right)\dot{\theta}^2 \tag{11.30}$$

$$U = -m_1 g b\cos\theta - m_2 g\frac{b}{2}\cos\theta \tag{11.31}$$

Where $U = 0$ at the origin. The equation of motion (Equation 11.28) follows directly from a straightforward application of the Lagrangian technique.

11.4 PRINCIPAL AXES OF INERTIA*

It should be clear that a considerable simplification in the expressions for T and $\mathbf{L}$ would result if the inertia tensor consisted only of diagonal elements. If we could write

$$I_{ij} = I_i\delta_{ij} \tag{11.32}$$

then the inertia tensor would be

$$\{\mathbf{I}\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix} \tag{11.33}$$

We would then have

$$L_i = \sum_j I_i\delta_{ij}\omega_j = I_i\omega_i \tag{11.34}$$

and

$$T_{\text{rot}} = \tfrac{1}{2}\sum_{i,j} I_i\delta_{ij}\omega_i\omega_j = \tfrac{1}{2}\sum_i I_i\omega_i^2 \tag{11.35}$$

Thus, the condition that $\{\mathbf{I}\}$ have only diagonal elements provides quite simple expressions for the angular momentum and the rotational kinetic energy. We now determine the conditions under which Equation 11.32 becomes the description of the inertia tensor. This involves finding a set of body axes for which the products of inertia (i.e., the off-diagonal elements of $\{\mathbf{I}\}$) vanish. We call such axes the **principal axes of inertia.**

If a body rotates around a principal axis, both the angular velocity and the angular momentum are, according to Equation 11.34, directed along this axis. Then, if I is the moment of inertia about this axis, we can write

$$\mathbf{L} = I\boldsymbol{\omega} \tag{11.36}$$

Equating the components of $\mathbf{L}$ in Equations 11.20a and 11.36, we have

$$\left.\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\} \tag{11.37}$$

Or, collecting terms, we obtain

$$\left.\begin{aligned} (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 &= 0 \end{aligned}\right\} \tag{11.38}$$

The condition that these equations have a nontrivial solution is that the determinant of the coefficients vanish:

$$\begin{vmatrix} (I_{11} - I) & I_{12} & I_{13} \\ I_{21} & (I_{22} - I) & I_{23} \\ I_{31} & I_{32} & (I_{33} - I) \end{vmatrix} = 0 \tag{11.39}$$

* Discovered by Euler in 1750.

The expansion of this determinant leads to the **secular equation*** for $\mathbf{I}$, which is a cubic. Each of the three roots corresponds to a moment of inertia about one of the principal axes. These values, I_1, I_2, and I_3, are called the **principal moments of inertia**. If the body rotates about the axis corresponding to the principal moment I_1, then Equation 11.36 becomes $\mathbf{L} = I_1\boldsymbol{\omega}$—that is, both $\boldsymbol{\omega}$ and $\mathbf{L}$ are directed along this axis. The direction of $\boldsymbol{\omega}$ with respect to the body coordinate system is then the same as the direction of the principal axis corresponding to I_1. Therefore, we can determine the direction of this principal axis by substituting I_1 for I in Equation 11.38 and determining the ratios of the components of the angular-velocity vector: $\omega_1 : \omega_2 : \omega_3$. We thereby determine the direction cosines of the axis about which the moment of inertia is I_1. The directions corresponding to I_2 and I_3 can be found in a similar fashion. That the principal axes determined in this manner are indeed *real* and *orthogonal* is proved in Section 11.6; these results also follow from the more general considerations given in Section 12.4.

The fact that the diagonalization procedure just described yields only the *ratios* of the components of $\boldsymbol{\omega}$ is no handicap, because the ratios completely determine the direction of each of the principal axes, and it is only the directions of these axes that is required. Indeed, we would not expect the *magnitudes* of the ω_i to be determined, because the actual rate of the body's angular motion cannot be specified by the geometry alone. We are free to impress on the body any magnitude of the angular velocity we wish.

For most of the problems encountered in rigid-body dynamics, the bodies are of some regular shape, so we can determine the principal axes merely by examining the symmetry of the body. For example, any body that is a solid of revolution (e.g., a cylindrical rod) has one principal axis that lies along the symmetry axis (e.g., the center line of the cylindrical rod), and the other two axes are in a plane perpendicular to the symmetry axis. It should be obvious that because the body is symmetrical, the choice of the angular placement of these other two axes is arbitrary. If the moment of inertia along the symmetry axis is I_1, then $I_2 = I_3$ for a solid of revolution—that is, the secular equation has a double root.

If a body has $I_1 = I_2 = I_3$, it is termed a **spherical top**; if $I_1 = I_2 \neq I_3$, it is termed a **symmetric top**; if the principal moments of inertia are all distinct, it is termed an **asymmetric top**. If a body has $I_1 = 0$, $I_2 = I_3$, as, for example, two point masses connected by a weightless shaft, or a diatomic molecule, it is called a **rotor**.

EXAMPLE 11.3

Find the principal moments of inertia and the principal axes for the cube in Example 11.1.

* So called because a similar equation describes secular perturbations in celestial mechanics. The mathematical terminology is the *characteristic polynomial.*

Solution: In Example 11.1, we found that the moment-of-inertia tensor for a cube (with origin at one corner) had nonzero off-diagonal elements. Evidently, the coordinate axes chosen for that calculation were not principal axes. If, for example, the cube rotates about the x_3-axis, then $\boldsymbol{\omega} = \omega_3\mathbf{e}_3$ and the angular momentum vector $\mathbf{L}$ (see Equation 11.37) has the components

$$L_1 = -\tfrac{1}{4}\beta\omega_3$$
$$L_2 = -\tfrac{1}{4}\beta\omega_3$$
$$L_3 = \tfrac{2}{3}\beta\omega_3$$

Thus,

$$\mathbf{L} = Mb^2\omega_3\left(-\tfrac{1}{4}\mathbf{e}_1 - \tfrac{1}{4}\mathbf{e}_2 + \tfrac{2}{3}\mathbf{e}_3\right)$$

which is not in the same direction as $\boldsymbol{\omega}$.

To find the principal moments of inertia, we must solve the secular equation

$$\begin{vmatrix} \tfrac{2}{3}\beta - I & -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta \\ -\tfrac{1}{4}\beta & \tfrac{2}{3}\beta - I & -\tfrac{1}{4}\beta \\ -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta & \tfrac{2}{3}\beta - I \end{vmatrix} = 0 \tag{11.40}$$

The value of a determinant is not affected by adding (or subtracting) any row (or column) from any other row (or column). Equation 11.40 can be solved more easily if we subtract the first row from the second:

$$\begin{vmatrix} \tfrac{2}{3}\beta - I & -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta \\ -\tfrac{11}{12}\beta + I & \tfrac{11}{12}\beta - I & 0 \\ -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta & \tfrac{2}{3}\beta - I \end{vmatrix} = 0$$

We can factor $(\tfrac{11}{12}\beta - I)$ from the second row:

$$\left(\tfrac{11}{12}\beta - I\right)\begin{vmatrix} \tfrac{2}{3}\beta - I & -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta \\ -1 & 1 & 0 \\ -\tfrac{1}{4}\beta & -\tfrac{1}{4}\beta & \tfrac{2}{3}\beta - I \end{vmatrix} = 0$$

Expanding, we have

$$\left(\tfrac{11}{12}\beta - I\right)\left[\left(\tfrac{2}{3}\beta - I\right)^2 - \tfrac{1}{8}\beta^2 - \tfrac{1}{4}\beta\left(\tfrac{2}{3}\beta - I\right)\right] = 0$$

which can be factored to obtain

$$(\tfrac{1}{6}\beta - I)(\tfrac{11}{12}\beta - I)(\tfrac{11}{12}\beta - I) = 0$$

Thus, we have the following roots, which give the principal moments of inertia:

$$I_1 = \tfrac{1}{6}\beta, \qquad I_2 = \tfrac{11}{12}\beta, \qquad I_3 = \tfrac{11}{12}\beta$$

The diagonalized moment-of-inertia tensor becomes

$$\{\mathbf{I}\} = \begin{Bmatrix} \tfrac{1}{6}\beta & 0 & 0 \\ 0 & \tfrac{11}{12}\beta & 0 \\ 0 & 0 & \tfrac{11}{12}\beta \end{Bmatrix} \tag{11.41}$$

Because two of the roots are identical, $I_2 = I_3$, the principal axis associated with I_1 must be an axis of symmetry.

To find the direction of the principal axis associated with I_1, we substitute for I in Equation 11.38 the value $I = I_1 = \tfrac{1}{6}\beta$:

$$\left.\begin{aligned} (\tfrac{2}{3}\beta - \tfrac{1}{6}\beta)\omega_{11} - \tfrac{1}{4}\beta\omega_{21} - \tfrac{1}{4}\beta\omega_{31} &= 0 \\ -\tfrac{1}{4}\beta\omega_{11} + (\tfrac{2}{3}\beta - \tfrac{1}{6}\beta)\omega_{21} - \tfrac{1}{4}\beta\omega_{31} &= 0 \\ -\tfrac{1}{4}\beta\omega_{11} - \tfrac{1}{4}\beta\omega_{21} + (\tfrac{2}{3}\beta - \tfrac{1}{6}\beta)\omega_{31} &= 0 \end{aligned}\right\}$$

where the second subscript 1 on the ω_i signifies that we are considering the principal axis associated with I_1. Dividing the first two of these equations by $\beta/4$, we have

$$\left.\begin{aligned} 2\omega_{11} - \omega_{21} - \omega_{31} &= 0 \\ -\omega_{11} + 2\omega_{21} - \omega_{31} &= 0 \end{aligned}\right\} \tag{11.42}$$

Subtracting the second of these equations from the first, we find $\omega_{11} = \omega_{21}$. Using this result in either of the Equations 11.42, we obtain $\omega_{11} = \omega_{21} = \omega_{31}$, and the desired ratios are

$$\omega_{11}:\omega_{21}:\omega_{31} = 1:1:1$$

Therefore, when the cube rotates about an axis that has associated with it the moment of inertia $I_1 = \tfrac{1}{6}\beta = \tfrac{1}{6}Mb^2$, the projections of $\boldsymbol{\omega}$ on the three coordinate axes are all equal. Hence, this principal axis corresponds to the diagonal of the cube.

Because the moments I_2 and I_3 are equal, the orientation of the principal axes associated with these moments is arbitrary; they need only lie in a plane normal to the diagonal of the cube.

11.5 MOMENTS OF INERTIA FOR DIFFERENT BODY COORDINATE SYSTEMS

For the kinetic energy to be separable into translational and rotational portions (see Equation 11.6), it is, in general, necessary to choose a body coordinate system whose origin is the center of mass of the body. For certain geometrical shapes, it may not always be convenient to compute the elements of the inertia tensor using such a coordinate system. We therefore consider some other set of coordinate axes X_i, also fixed with respect to the body and having the same orientation as the x_i-axes but with an origin Q that does not correspond with the origin O (located at the center of mass of the body coordinate system). Origin Q may be located either within or outside the body under consideration.

The elements of the inertia tensor relative to the X_i-axes can be written as

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i}X_{\alpha,j}\right) \tag{11.43}$$

If the vector connecting Q with O is $\mathbf{a}$, then the general vector $\mathbf{R}$ (Figure 11-4) can be written as

$$\mathbf{R} = \mathbf{a} + \mathbf{r} \tag{11.44}$$

with components

$$X_i = a_i + x_i \tag{11.45}$$

Using Equation 11.45, the tensor element J_{ij} becomes

$$\begin{aligned} J_{ij} &= \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k (x_{\alpha,k} + a_k)^2 - (x_{\alpha,i} + a_i)(x_{\alpha,j} + a_j)\right) \\ &= \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right) \end{aligned}$$

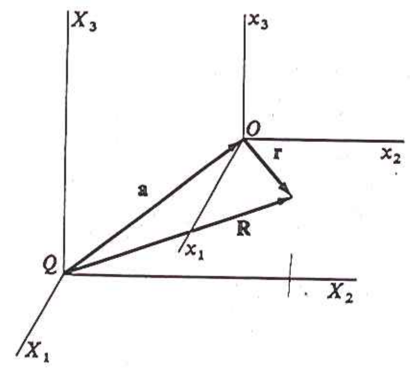


FIGURE 11-4

$$+\sum_\alpha m_\alpha\left(\delta_{ij}\sum_k(2x_{\alpha,k}a_k + a_k^2) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j)\right) \tag{11.46}$$

Identifying the first summation as I_{ij}, we have, on regrouping,

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha\left(\delta_{ij}\sum_k a_k^2 - a_i a_j\right)$$
$$+\sum_\alpha m_\alpha\left(2\delta_{ij}\sum_k x_{\alpha,k}a_k - a_i x_{\alpha,i} - a_j x_{\alpha,i}\right) \tag{11.47}$$

But each term in the last summation involves a sum of the form

$$\sum_\alpha m_\alpha x_{\alpha,k}$$

We know, however, that because O is located at the center of mass,

$$\sum_\alpha m_\alpha \mathbf{r}_\alpha = 0\ .$$

or, for the kth component,

$$\sum_\alpha m_\alpha x_{\alpha,k} = 0$$

Therefore, all such terms in Equation 11.47 vanish and we have

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha\left(\delta_{ij}\sum_k a_k^2 - a_i a_j\right) \tag{11.48}$$

But

$$\sum_\alpha m_\alpha = M \quad \text{and} \quad \sum_k a_k^2 \equiv a^2$$

Solving for I_{ij}, we have the result

$$\boxed{I_{ij} = J_{ij} - M(a^2\delta_{ij} - a_i a_j)} \tag{11.49}$$

which allows the calculation of the elements I_{ij} of the desired inertia tensor (with origin at the center of mass) once those with respect to the X_i-axes are known. The second term on the right-hand side of Equation 11.49 is the inertia tensor referred to the origin Q for a point mass M.

Equation 11.49 is the general form of **Steiner's parallel-axis theorem,*** the simplified form of which is given in elementary treatments. Consider, for example,

* Jacob Steiner (1796–1863).

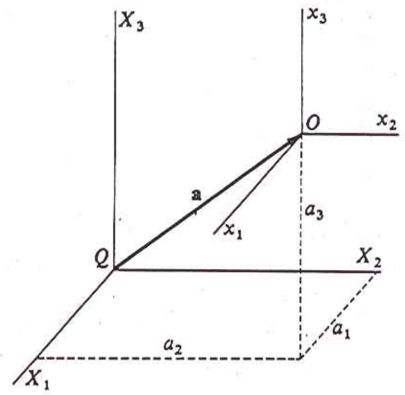


FIGURE 11-5

Figure 11-5. Element I_{11} is

$$I_{11} = J_{11} - M[(a_1^2 + a_2^2 + a_3^2)\,\delta_{11} - a_1^2]$$
$$= J_{11} - M(a_2^2 + a_3^2)$$

which states that the difference between the elements is equal to the mass of the body multiplied by the square of the distance between the parallel axes (in this case, between the x_1- and X_1-axes).

EXAMPLE 11.4

Find the inertia tensor of the cube of Example 11.1 in a coordinate system with origin at the center of mass.

Solution: In Example 11.1, with the origin at the corner of the cube, we found the inertia tensor to be

$$\{\mathbf{J}\} = \begin{Bmatrix} \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 \end{Bmatrix} \tag{11.50}$$

We may now use Equation 11.49 to obtain the inertia tensor $\{\mathbf{I}\}$ referred to a coordinate system with origin at the center of mass. In keeping with the notation of this section, we call the new axes x_i with origin O and call the previous axes X_i with origin Q at one corner of the cube (Figure 11-6).

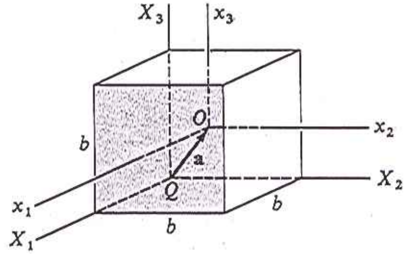


FIGURE 11-6

The center of mass of the cube is at the point $(b/2, b/2, b/2)$ in the X_i coordinate system, and the components of the vector **a** therefore are

$$a_1 = a_2 = a_3 = b/2$$

From Equation 11.50, we have

$$\left.\begin{aligned} J_{11} &= J_{22} = J_{33} = \tfrac{2}{3}Mb^2 \\ J_{12} &= J_{13} = J_{23} = -\tfrac{1}{4}Mb^2 \end{aligned}\right\}$$

And applying Equation 11.49, we find

$$\begin{aligned} I_{11} &= J_{11} - M(a^2 - a_1^2) \\ &= J_{11} - M(a_2^2 + a_3^2) \\ &= \tfrac{2}{3}Mb^2 - \tfrac{1}{2}Mb^2 = \tfrac{1}{6}Mb^2 \end{aligned}$$

and

$$\begin{aligned} I_{12} &= J_{12} - M(-a_1 a_2) \\ &= -\tfrac{1}{4}Mb^2 + \tfrac{1}{4}Mb^2 = 0 \end{aligned}$$

Altogether, we have

$$I_{12} = I_{22} = I_{33} = \tfrac{1}{6}Mb^2$$

$$I_{12} = I_{13} = I_{23} = 0$$

The inertia tensor is therefore diagonal:

$$\{\mathbf{I}\} = \begin{Bmatrix} \tfrac{1}{6}Mb^2 & 0 & 0 \\ 0 & \tfrac{1}{6}Mb^2 & 0 \\ 0 & 0 & \tfrac{1}{6}Mb^2 \end{Bmatrix} \tag{11.51}$$

If we factor out the common term $\frac{1}{6}Mb^2$ from this expression, we can write

$$\{\mathbf{I}\} = \tfrac{1}{6}Mb^2\{\mathbf{1}\} \tag{11.52}$$

where $\{\mathbf{1}\}$ is the **unit tensor:**

$$\{\mathbf{1}\} \equiv \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \tag{11.53}$$

Thus, we find that, for the choice of the origin at the center of mass of the cube, the principal axes are perpendicular to the faces of the cube. Because, from a physical standpoint, nothing distinguishes any one of these axes from another, the principal moments of inertia are all equal for this case. We note further that, as long as we maintain the origin at the center of mass, then the inertia tensor is the same for *any* orientation of the coordinate axes and these axes are equally valid principal axes.*

11.6 FURTHER PROPERTIES OF THE INERTIA TENSOR

Before attacking the problems of rigid-body dynamics by obtaining the general equations of motion, we should consider the fundamental importance of some of the operations we have been discussing. Let us begin by examining the properties of the inertia tensor under coordinate transformations.†

We have already obtained the fundamental relation connecting the inertia tensor and the angular momentum and angular velocity vectors (Equation 11.20), which we can write as

$$L_k = \sum_l I_{kl}\omega_l \tag{11.54a}$$

Because this is a vector equation, in a coordinate system rotated with respect to the system for which Equation 11.54a applies, we must have an entirely analogous relation,

$$L_i' = \sum_j I_{ij}'\omega_j' \tag{11.54b}$$

where the primed quantities all refer to the rotated system. Both **L** and **ω** obey the standard transformation equation for vectors (Equation 1.8):

$$x_i = \sum_j \lambda_{ij}^t x_j' = \sum_j \lambda_{ji} x_j'$$

* In this regard, the cube is similar to a sphere as far as the inertia tensor is concerned (i.e., for an origin at the center of mass, the structure of the inertia tensor elements is not sufficiently detailed to discriminate between a cube and a sphere).

† We confine our attention to rectangular coordinate systems so that we may ignore some of the more complicated properties of tensors that manifest themselves in general curvilinear coordinates.

We can therefore write

$$L_k = \sum_m \lambda_{mk} L'_m \qquad \text{(11.55a)}$$

and

$$\omega_l = \sum_j \lambda_{jl} \omega'_j \qquad \text{(11.55b)}$$

If we substitute Equations 11.55a and b into Equation 11.54a, we obtain

$$\sum_m \lambda_{mk} L'_m = \sum_l I_{kl} \sum_j \lambda_{jl} \omega'_j \qquad \text{(11.56)}$$

Next, we multiply both sides of this equation by λ_{ik} and sum over k:

$$\sum_m \left(\sum_k \lambda_{ik} \lambda_{mk} \right) L'_m = \sum_j \left(\sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \right) \omega'_j \qquad \text{(11.57)}$$

The term in parentheses on the left-hand side is just δ_{im}, so performing the summation over m we obtain

$$L'_i = \sum_j \left(\sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \right) \omega'_j \qquad \text{(11.58)}$$

For this equation to be identical with Equation 11.54b, we must have

$$I'_{ij} = \sum_{k,l} \lambda_{ik} \lambda_{jl} I_{kl} \qquad \text{(11.59)}$$

This is therefore the rule that the inertia tensor must obey under a coordinate transformation. Equation 11.59 is, in fact, the *general* rule specifying the manner in which any second-rank tensor must transform. For a tensor $\{\mathbf{T}\}$ of arbitrary rank, the statement is*

$$T'_{abcd\ldots} = \sum_{i,j,k,l,\ldots} \lambda_{ai} \lambda_{bj} \lambda_{ck} \lambda_{dl} \cdots T_{ijkl\ldots} \qquad \text{(11.60)}$$

Note that we can write Equation 11.59 as

$$\boxed{I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^t_{lj}} \qquad \text{(11.61)}$$

Although matrices and tensors are distinct types of mathematical objects, the manipulation of tensors is in many respects the same as for matrices. Thus, Equation 11.61 can be expressed as a matrix equation:

$$\mathbf{I}' = \boldsymbol{\lambda} \mathbf{I} \boldsymbol{\lambda}^t \qquad \text{(11.62)}$$

where we understand $\mathbf{I}$ to be the matrix consisting of the elements of the tensor $\{\mathbf{I}\}$. Because we are considering only orthogonal transformation matrices, the transpose of $\boldsymbol{\lambda}$ is equal to its inverse, so we can express Equation 11.62 as

$$\boxed{\mathbf{I}' = \boldsymbol{\lambda} \mathbf{I} \boldsymbol{\lambda}^{-1}} \qquad \text{(11.63)}$$

A transformation of this general type is called a **similarity transformation** ($\mathbf{I}'$ is similar to $\mathbf{I}$).

* Note that a tensor of the *first rank* transforms as

$$T'_a = \sum_i \lambda_{ai} T_i$$

Such a tensor is in fact a *vector*. A tensor of *zero* rank implies that $T' = T$, or that such a tensor is a scalar. The properties of quantities that transform in this manner were first discussed by C. Niven in 1874. The application of the term *tensor* to such quantities can be traced to J. Willard Gibbs.

EXAMPLE 11.5

Prove the assertion stated in Example 11.4 that the inertia tensor for a cube (with origin at the center of mass) is independent of the orientation of the axes.

Solution: The change in the inertia tensor under a rotation of the coordinate axes can be computed by making a similarity transformation. Thus, if the rotation is described by the matrix $\boldsymbol{\lambda}$, we have

$$\mathbf{I}' = \boldsymbol{\lambda} \mathbf{I} \boldsymbol{\lambda}^{-1} \qquad \text{(11.64)}$$

But the matrix $\mathbf{I}$, which is derived from the elements of the tensor $\{\mathbf{I}\}$ (Equation 11.52 of Example 11.4), is just the identity matrix $\mathbf{1}$ multiplied by a constant:

$$\mathbf{I} = \tfrac{1}{6} Mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tfrac{1}{6} Mb^2 \mathbf{1} \qquad \text{(11.65)}$$

Therefore, the operations specified in Equation 11.64 are trivial:

$$\mathbf{I}' = \tfrac{1}{6} Mb^2 \boldsymbol{\lambda} \mathbf{1} \boldsymbol{\lambda}^{-1} = \tfrac{1}{6} Mb^2 \boldsymbol{\lambda}\boldsymbol{\lambda}^{-1} = \tfrac{1}{6} Mb^2 \mathbf{1} = \mathbf{I} \qquad \text{(11.66)}$$

Thus, the transformed inertia tensor is identical to the original tensor, independent of the details of the rotation.

Let us next determine what condition must be satisfied if we take an arbitrary inertia tensor and perform a coordinate rotation in such a way that the transformed inertia tensor is diagonal. Such an operation implies that the quantity I'_{ij} in Equation 11.59 must satisfy (see Equation 11.32) the relation

$$I'_{ij} = I_i \delta_{ij} \qquad \text{(11.67)}$$

Thus,

$$I_i\delta_{ij} = \sum_{k,l} \lambda_{ik}\lambda_{jl}I_{kl} \tag{11.68}$$

If we multiply both sides of this equation by λ_{im} and sum over i, we obtain

$$\sum_i I_i\lambda_{im}\delta_{ij} = \sum_{k,l}\left(\sum_i \lambda_{im}\lambda_{ik}\right)\lambda_{jl}I_{kl} \tag{11.69}$$

The term in parentheses is just δ_{mk}, so the summation over i on the left-hand side of the equation and the summation over k on the right-hand side yield

$$I_j\lambda_{jm} = \sum_l \lambda_{jl}I_{ml} \tag{11.70}$$

Now the left-hand side of this equation can be written as

$$I_j\lambda_{jm} = \sum_l I_j\lambda_{jl}\delta_{ml} \tag{11.71}$$

so Equation 11.70 becomes

$$\sum_l I_j\lambda_{jl}\delta_{ml} = \sum_l \lambda_{jl}I_{ml} \tag{11.72a}$$

or

$$\sum_l (I_{ml} - I_j\delta_{ml})\lambda_{jl} = 0 \tag{11.72b}$$

This is a set of simultaneous linear algebraic equations; for each value of j there are three such equations, one for each of the three possible values of m. For a nontrivial solution to exist, the determinant of the coefficients must vanish, so the principal moments of inertia, I_1, I_2, and I_3, are obtained as roots of the secular determinant for I:

$$\boxed{|I_{ml} - I\delta_{ml}| = 0} \tag{11.73}$$

This equation is just Equation 11.39; it is a cubic equation that yields the principal moments of inertia.

Thus, for any inertia tensor, the elements of which are computed for a given origin, it is possible to perform a rotation of the coordinate axes about that origin in such a way that the inertia tensor becomes diagonal. The new coordinate axes are then the principal axes of the body, and the new moments are the principal moments of inertia. Thus, for any body and for any choice of origin, there always exists a set of principal axes.

EXAMPLE 11.6

For the cube of Example 11.1, diagonalize the inertia tensor by rotating the coordinate axes.

Solution: We choose the origin to lie at one corner and perform the rotation in such a manner that the x_1-axis coincides with the diagonal of the cube. Such a rotation can conveniently be made in two steps: first, we rotate through an angle of 45° about the x_3-axis; second, we rotate through an angle of $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$ about the x_2'-axis. The first rotation matrix is

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.74}$$

and the second rotation matrix is

$$\boldsymbol{\lambda}_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \tag{11.75}$$

The complete rotation matrix is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix} \tag{11.76}$$

The matrix form of the transformed inertia tensor (see Equation 11.62) is

$$\mathbf{I}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^t \tag{11.77}$$

or, factoring β out of $\mathbf{I}$,

$$\mathbf{I}' = \frac{\beta}{3}\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & \sqrt{2} \end{pmatrix}$$

$$\mathbf{I}' = \frac{\beta}{3}\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{6} & -\frac{11}{12}\sqrt{\frac{3}{2}} & -\frac{11}{12}\frac{\sqrt{2}}{2} \\ \frac{1}{6} & \frac{11}{12}\sqrt{\frac{3}{2}} & -\frac{11}{12}\frac{\sqrt{2}}{2} \\ \frac{1}{6} & 0 & \frac{11}{12}\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{pmatrix} \tag{11.78}$$

Equation 11.78 is just the matrix form of the inertia tensor found by the diagonalization procedure using the secular determinant (Equation 11.41 of Example 11.3).

We have demonstrated two general procedures to diagonalize the inertia tensor. We previously pointed out that these methods are not limited to the inertia tensor but are generally valid. Either procedure can be very complicated. For example, if we wish to use the rotation procedure in the most general case, we must first construct a matrix that describes an arbitrary rotation. This entails three separate rotations, one about each of the coordinate axes. This rotation matrix must then be applied to the tensor in a similarity transformation. The off-diagonal elements of the resulting matrix* must then be examined and values of the rotation angles determined so that these off-diagonal elements vanish. The actual use of such a procedure can tax the limits of human patience, but in some simple situations, this method of diagonalization can be used with profit. This is particularly true if the geometry of the problem indicates that only a simple rotation about one of the coordinate axes is necessary; the rotation angle can then be evaluated without difficulty (see, for example, Problems 11-16, 11-18, and 11-19).

In practice, there are systematic procedures for finding principal moments and principal axes of any inertia tensor. Standard computer programs and hand-calculator methods are available to find the n roots of an nth-order polynomial and to diagonalize a matrix. When the principal moments are known, the principal axes are easily found.

* A *large* sheet of paper should be used!

The example of the cube illustrates the important point that the elements of the inertia tensor, the values of the principal moments of inertia, and the orientation of the principal axes for a rigid body all depend on the choice of origin for the system. Recall, however, that for the kinetic energy to be separable into translational and rotational portions, the origin of the body coordinate system must, in general, be taken to coincide with the center of mass of the body. However, for *any* choice of the origin for *any* body, there always exists an orientation of the axes that diagonalizes the inertia tensor. Hence, these axes become principal axes for that particular origin.

Next, we seek to prove that the principal axes actually form an orthogonal set. Let us assume that we have solved the secular equation and have determined the principal moments of inertia, all of which are distinct. We know that for each principal moment there exists a corresponding principal axis with the property that, if the angular velocity vector $\boldsymbol{\omega}$ lies along this axis, then the angular momentum vector $\mathbf{L}$ is similarly oriented; that is, to each I_j there corresponds an angular velocity $\boldsymbol{\omega}_j$ with components ω_{1j}, ω_{2j}, ω_{3j}. (We use the subscript on the vector $\boldsymbol{\omega}$ and the second subscript on the components of $\boldsymbol{\omega}$ to designate the principal moment with which we are concerned.) For the mth principal moment, we have

$$L_{im} = I_m \omega_{im} \tag{11.79}$$

In terms of the elements of the moment-of-inertia tensor, we also have

$$L_{im} = \sum_k I_{ik}\omega_{km} \tag{11.80}$$

Combining these two relations, we have

$$\sum_k I_{ik}\omega_{km} = I_m \omega_{im} \tag{11.81a}$$

Similarly, we can write for the nth principal moment:

$$\sum_i I_{ki}\omega_{in} = I_n \omega_{kn} \tag{11.81b}$$

If we multiply Equation 11.81a by ω_{in} and sum over i and then multiply Equation 11.81b by ω_{km} and sum over k, we have

$$\left.\begin{aligned} \sum_{i,k} I_{ik}\omega_{km}\omega_{in} &= \sum_i I_m \omega_{im}\omega_{in} \\ \sum_{i,k} I_{ki}\omega_{in}\omega_{km} &= \sum_k I_n \omega_{kn}\omega_{km} \end{aligned}\right\} \tag{11.82}$$

The left-hand sides of these equations are identical, because the inertia tensor is symmetrical ($I_{ik} = I_{ki}$). Therefore, on subtracting the second equation from the first, we have

$$I_m \sum_i \omega_{im}\omega_{in} - I_n \sum_k \omega_{km}\omega_{kn} = 0 \tag{11.83}$$

Because i and k are both dummy indices, we can replace them by l, say, and obtain

$$(I_m - I_n)\sum_l \omega_{lm}\omega_{ln} = 0 \tag{11.84}$$

By hypothesis, the principal moments are distinct, so that $I_m \neq I_n$. Therefore, Equation 11.84 can be satisfied only if

$$\sum_l \omega_{lm}\omega_{ln} = 0 \tag{11.85}$$

But this summation is just the definition of the scalar product of the vectors $\boldsymbol{\omega}_m$ and $\boldsymbol{\omega}_n$. Hence,

$$\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0 \tag{11.86}$$

Because the principal moments I_m and I_n were picked arbitrarily from the set of three moments, we conclude that each pair of principal axes is perpendicular; the three principal axes therefore constitute an orthogonal set.

If a double root of the secular equation exists, so that the principal moments are I_1, $I_2 = I_3$, then the preceding analysis shows that the angular velocity vectors satisfy the relations

$$\boldsymbol{\omega}_1 \perp \boldsymbol{\omega}_2, \qquad \boldsymbol{\omega}_1 \perp \boldsymbol{\omega}_3$$

but that nothing may be said regarding the angle between $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$. But the fact that $I_2 = I_3$ implies that the body possesses an axis of symmetry. Therefore, $\boldsymbol{\omega}_1$ lies along the symmetry axis; and $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ are required only to lie in the plane perpendicular to $\boldsymbol{\omega}_1$. Consequently, there is no loss of generality if we also choose $\boldsymbol{\omega}_2 \perp \boldsymbol{\omega}_3$. Thus, the principal axes for a rigid body with an axis of symmetry can also be chosen to be an orthogonal set.

We have previously shown that the principal moments of inertia are obtained as the roots of the secular equation—a cubic equation. Mathematically, at least one of the roots of a cubic equation must be real, but there may be two imaginary roots. If the diagonalization procedures for the inertia tensor are to be physically meaningful, we must always obtain only real values for the principal moments. We can show in the following way that this is a general result. First, we assume the roots to be complex and use a procedure similar to that used in the preceding proof. But now we must also allow the quantities ω_{km} to become complex. There is no mathematical reason why we cannot do this, and we are not concerned with any physical interpretation of these quantities. We therefore write Equation 11.81a as before, but we take the complex conjugate of Equation 11.81b:

$$\left.\begin{aligned} \sum_k I_{ik}\omega_{km} &= I_m\omega_{im} \\ \sum_i I^*_{ki}\omega^*_{in} &= I^*_n\omega^*_{kn} \end{aligned}\right\} \tag{11.87}$$

Next, we multiply the first of these equations by ω^*_{in} and sum over i and multiply the second by ω_{km} and sum over k. The inertia tensor is symmetrical, and its elements are all real, so that $I_{ik} = I^*_{ki}$. Therefore, subtracting the second of these equations from the first, we find

$$(I_m - I^*_n) \sum_l \omega_{lm}\omega^*_{ln} = 0 \tag{11.88}$$

For the case $m = n$, we have

$$(I_m - I^*_m) \sum_l \omega_{lm}\omega^*_{lm} = 0 \tag{11.89}$$

The sum is just the definition of the scalar product of $\boldsymbol{\omega}_m$ and $\boldsymbol{\omega}^*_m$:

$$\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}^*_m = |\boldsymbol{\omega}_m|^2 \geq 0 \tag{11.90}$$

Therefore, because the squared magnitude of $\boldsymbol{\omega}_m$ is in general positive, it must be true that $I_m = I^*_m$ for Equation 11.89 to be satisfied. If a quantity and its complex conjugate are equal, then the imaginary parts must vanish identically. Thus, the principal moments of inertia are all real. Because $\{\mathbf{I}\}$ is real, the vectors $\boldsymbol{\omega}_m$ must also be real.

If $m \neq n$ in Equation 11.88 and if $I_m \neq I_n$, then the equation can be satisfied only if $\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0$; that is, these vectors are orthogonal, as before.

In all the proofs carried out in this section, we have referred to the inertia tensor. But examining these proofs reveals that the only properties of the inertia tensor that have actually been used are the facts that the tensor is symmetrical and that the elements are real. We may therefore conclude that *any* real, symmetric tensor* has the following properties:

1. Diagonalization may be accomplished by an appropriate rotation of axes, that is, a similarity transformation.
2. The eigenvalues† are obtained as roots of the secular determinant and are real.
3. The eigenvectors† are real and orthogonal.

11.7 EULERIAN ANGLES

The transformation from one coordinate system to another can be represented by a matrix equation of the form

$$\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}'$$

If we identify the fixed system with $\mathbf{x}'$ and the body system with $\mathbf{x}$, then the rotation matrix $\boldsymbol{\lambda}$ completely describes the relative orientation of the two systems. The rotation matrix $\boldsymbol{\lambda}$ contains three independent angles. There are many possible choices for these angles; we find it convenient to use the **Eulerian angles**‡ ϕ, θ, and ψ.

* To be more precise, we require only that the elements of the tensor obey the relation $I_{ik} = I^*_{ki}$; thus we allow the possibility of complex quantities. Tensors (and matrices) with this property are said to be **Hermitean.**

† The terms *eigenvalues* and *eigenvectors* are the generic names of the quantities, which, in the case of the inertia tensor, are the principal moments and the principal axes, respectively. We shall encounter these terms again in the discussion of small oscillations in Chapter 12.

‡ The rotation scheme of Euler was first published in 1776.

The Eulerian angles are generated in the following series of rotations, which takes the x_i' system into the x_i system*:

1. The first rotation is counterclockwise through an angle ϕ about the x_3'-axis (Figure 11-7a) to transform the x_i' into the x_i''. Because the rotation takes place in the x_1'-x_2' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.91}$$

and

$$\mathbf{x}'' = \boldsymbol{\lambda}_\phi \mathbf{x}' \tag{11.92}$$

2. The second rotation is counterclockwise through an angle θ about the x_1''-axis (Figure 11-7b) to transform the x_i'' into the x_i'''. Because the rotation is now

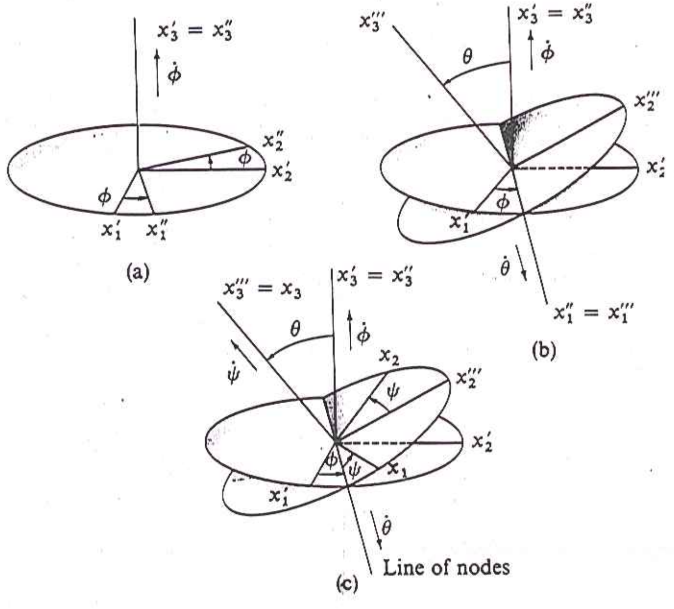


FIGURE 11-7

* The designations of the Euler angles and even the manner in which they are generated are not universally agreed upon. Therefore, some care must be taken in comparing any results from different sources. The notation used here is that most commonly found in modern texts.

in the x_2''-x_3'' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{11.93}$$

and

$$\mathbf{x}''' = \boldsymbol{\lambda}_\theta \mathbf{x}'' \tag{11.94}$$

3. The third rotation is counterclockwise through an angle ψ about the x_3'''-axis (Figure 11-7c) to transform the x_i''' into the x_i. The transformation matrix is

$$\boldsymbol{\lambda}_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.95}$$

and

$$\mathbf{x} = \boldsymbol{\lambda}_\psi \mathbf{x}''' \tag{11.96}$$

The line common to the planes containing the x_1- and x_2-axes and the x_1'- and x_2'-axes is called the **line of nodes**. The complete transformation from the x_i' system to the x_i system is given by

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\lambda}_\psi \mathbf{x}''' = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \mathbf{x}'' \\ &= \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \mathbf{x}' \end{aligned} \tag{11.97}$$

and the rotation matrix $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \tag{11.98}$$

The components of this matrix are

$$\left.\begin{aligned} \lambda_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\ \lambda_{21} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\ \lambda_{31} &= \sin\theta\sin\phi \\ \lambda_{12} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\ \lambda_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\ \lambda_{32} &= -\sin\theta\cos\phi \\ \lambda_{13} &= \sin\psi\sin\theta \\ \lambda_{23} &= \cos\psi\sin\theta \\ \lambda_{33} &= \cos\theta \end{aligned}\right\} \tag{11.99}$$

(The components λ_{ij} are off-set in the preceding equation to assist in the visualization of the complete $\boldsymbol{\lambda}$ matrix.)

Because we can associate a vector with an infinitesimal rotation, we can associate the time derivatives of these rotation angles with the components of the angular velocity vector **ω**. Thus,

$$\left.\begin{aligned}\omega_\phi &= \dot\phi\\ \omega_\theta &= \dot\theta\\ \omega_\psi &= \dot\psi\end{aligned}\right\}\tag{11.100}$$

The rigid-body equations of motion are most conveniently expressed in the body coordinate system (i.e., the x_i system), and therefore we must express the components of **ω** in this system. We note that in Figure 11-7 the angular velocities $\dot{\boldsymbol\phi}$, $\dot{\boldsymbol\theta}$, and $\dot{\boldsymbol\psi}$ are directed along the following axes:

$\dot{\boldsymbol\phi}$ along the x_3'- (fixed) axis

$\dot{\boldsymbol\theta}$ along the line of nodes

$\dot{\boldsymbol\psi}$ along the x_3- (body) axis

The components of these angular velocities along the body coordinate axes are

$$\left.\begin{aligned}\dot\phi_1 &= \dot\phi\sin\theta\sin\psi\\ \dot\phi_2 &= \dot\phi\sin\theta\cos\psi\\ \dot\phi_3 &= \dot\phi\cos\theta\end{aligned}\right\}\tag{11.101a}$$

$$\left.\begin{aligned}\dot\theta_1 &= \dot\theta\cos\psi\\ \dot\theta_2 &= -\dot\theta\sin\psi\\ \dot\theta_3 &= 0\end{aligned}\right\}\tag{11.101b}$$

$$\left.\begin{aligned}\dot\psi_1 &= 0\\ \dot\psi_2 &= 0\\ \dot\psi_3 &= \dot\psi\end{aligned}\right\}\tag{11.101c}$$

Collecting the individual components of **ω**, we have, finally,

$$\boxed{\begin{aligned}\omega_1 &= \dot\phi_1+\dot\theta_1+\dot\psi_1 = \dot\phi\sin\theta\sin\psi+\dot\theta\cos\psi\\ \omega_2 &= \dot\phi_2+\dot\theta_2+\dot\psi_2 = \dot\phi\sin\theta\cos\psi-\dot\theta\sin\psi\\ \omega_3 &= \dot\phi_3+\dot\theta_3+\dot\psi_3 = \dot\phi\cos\theta+\dot\psi\end{aligned}}\tag{11.102}$$

These relations will be of use later in expressing the components of the angular momentum in the body coordinate system.

EXAMPLE 11.7

Using the Eulerian angles, find the transformation that moves the original x_1'-axis to the x_2'-x_3' plane halfway between x_2' and x_3' and moves x_2' perpendicular to the x_2'-x_3' plane (Figure 11-8).

Solution: The key to transformations using Eulerian angles is the second rotation about the line of nodes, because this single rotation must move x_3' to x_3. From the statement of the problem, x_3 must be in the x_2'-x_3' plane, rotated 45° from x_3'. The first rotation must move x_1' to x_1'' to have the correct position to rotate $x_3' = x_3''$ to $x_3''' = x_3$.

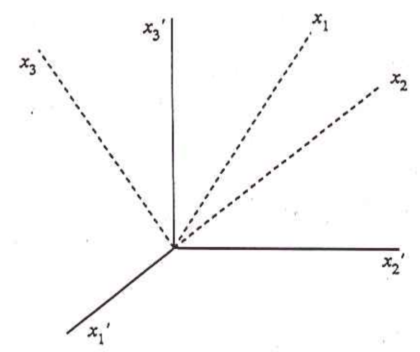


FIGURE 11-8

In this case, $x_3' = x_3''$ is rotated $\theta = 45°$ about the original $x_1' = x_1''$-axis so that $\phi = 0$ and

$$\boldsymbol\lambda_\phi = \mathbf{1}\tag{11.103}$$

$$\boldsymbol\lambda_\theta = \begin{pmatrix}1 & 0 & 0\\ 0 & 1/\sqrt2 & 1/\sqrt2\\ 0 & -1/\sqrt2 & 1/\sqrt2\end{pmatrix}\tag{11.104}$$

The last rotation, $\psi = 90°$, moves $x_1' = x_1'' = x_1'''$ to x_1 to the position desired in the original x_2'–x_3' plane.

$$\boldsymbol\lambda_\psi = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\tag{11.105}$$

The transformation matrix $\boldsymbol\lambda$ is $\boldsymbol\lambda = \boldsymbol\lambda_\psi\boldsymbol\lambda_\theta\boldsymbol\lambda_\phi = \boldsymbol\lambda_\psi\boldsymbol\lambda_\theta$:

$$\boldsymbol\lambda = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1/\sqrt2 & 1/\sqrt2\\ 0 & -1/\sqrt2 & 1/\sqrt2\end{pmatrix}$$

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad (11.106)$$

:tion comparison between the x_i- and x_i'-axes shows that $\boldsymbol{\lambda}$ represents a single on describing the transformation.

B EULER'S EQUATIONS FOR A RIGID BODY

:s first consider the force-free motion of a rigid body. In such a case, the poten- :nergy U vanishes and the Lagrangian L becomes identical with the rotational ic energy T.* If we choose the x_i-axes to correspond to the principal axes of body, then from Equation 11.35 we have

$$T = \frac{1}{2}\sum_i I_i \omega_i^2 \qquad (11.107)$$

e choose the Eulerian angles as the generalized coordinates, then Lagrange's tion for the coordinate ψ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} = 0 \qquad (11.108)$$

:h can be expressed as

$$\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = 0 \qquad (11.109)$$

e differentiate the components of $\boldsymbol{\omega}$ (Equation 11.102) with respect to ψ and ve have

$$\left.\begin{aligned} \frac{\partial \omega_1}{\partial \psi} &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \omega_2 \\ \frac{\partial \omega_2}{\partial \psi} &= -\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\omega_1 \\ \frac{\partial \omega_3}{\partial \psi} &= 0 \end{aligned}\right\} \qquad (11.110)$$

$$\left.\begin{aligned} \frac{\partial \omega_1}{\partial \dot{\psi}} &= \frac{\partial \omega_2}{\partial \dot{\psi}} = 0 \\ \frac{\partial \omega_3}{\partial \dot{\psi}} &= 1 \end{aligned}\right\} \qquad (11.111)$$

:cause the motion is force free, the translational kinetic energy is unimportant for our purposes here. can always transform to a coordinate system in which the center of mass of the body is at rest.)

From Equation 11.107, we also have

$$\frac{\partial T}{\partial \omega_i} = I_i \omega_i \qquad (11.112)$$

Equation 11.109 therefore becomes

$$I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{d}{dt}I_3\omega_3 = 0$$

or

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \qquad (11.113)$$

Because the designation of any particular principal axis as the x_3-axis is entirely arbitrary, Equation 11.113 can be permuted to obtain relations for $\dot{\omega}_1$ and $\dot{\omega}_2$:

$$\left.\begin{aligned} (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 &= 0 \\ (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 &= 0 \end{aligned}\right\} \qquad (11.114)$$

Equations 11.114 are called **Euler's equations** for force-free motion.* It must be noted that, although Equation 11.113 for $\dot{\omega}_3$ is indeed the Lagrange equation for the coordinate ψ, the Euler equations for $\dot{\omega}_1$ and $\dot{\omega}_2$ are *not* the Lagrange equations for θ and ϕ.

To obtain Euler's equations for motion in a force field, we may start with the fundamental relation (see Equation 2.83) for the torque $\mathbf{N}$:

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \mathbf{N} \qquad (11.115)$$

where the designation "fixed" has been explicitly appended to $\dot{\mathbf{L}}$ because this relation is derived from Newton's equation and is therefore valid only in an inertial frame of reference. From Equation 10.12, we have

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} \qquad (11.116)$$

or

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N} \qquad (11.117)$$

The component of this equation along the x_3-axis (note that this is a *body* axis) is

$$\dot{L}_3 + \omega_1 L_2 - \omega_2 L_1 = N_3 \qquad (11.118)$$

* Leonard Euler, 1758.

But because we have chosen the x_i-axes to coincide with the principal axes of the body, we have, from Equation 11.34,

$$L_i = I_i\omega_i$$

so that

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \tag{11.119}$$

By permuting the subscripts, we can write all three components of **N**:

$$\left.\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= N_3 \end{aligned}\right\} \tag{11.120}$$

Using the permutation symbol, we can write, in general

$$\boxed{(I_i - I_j)\omega_i\omega_j - \sum_k (I_k\dot{\omega}_k - N_k)\varepsilon_{ijk} = 0} \tag{11.121}$$

Equations 11.120 and 11.121 are the desired Euler equations for the motion of a rigid body in a force field.

The motion of a rigid body depends on the structure of the body only through the three numbers I_1, I_2, and I_3—that is, the principal moments of inertia. Thus, any two bodies with the same principal moments move in exactly the same manner, regardless of the fact that they may have quite different shapes. (However, effects such as frictional retardation may depend on the shape of a body.) The simplest geometrical shape that a body having three given principal moments may possess is a homogeneous ellipsoid. The motion of any rigid body can therefore be represented by the motion of the **equivalent ellipsoid.*** The treatment of rigid-body dynamics from this point of view was originated by Poinsot in 1834. The **Poinsot construction** is sometimes useful for depicting the motion of a rigid body geometrically.†

EXAMPLE 11.8

Consider the dumbbell of Section 11.3. Find the angular momentum of the system and the torque required to maintain the motion shown in Figures 11-2 and 11-9.

Solution: Let $|\mathbf{r}_1| = |\mathbf{r}_2| = b$. Let the body fixed coordinate system have its origin at O and the symmetry axis x_3 be along the weightless shaft toward m_1.

* The momental ellipsoid was introduced by the French mathematician Baron Augustin Louis Cauchy (1789–1857) in 1827.

† See, for example, Goldstein (Go80, p. 205).

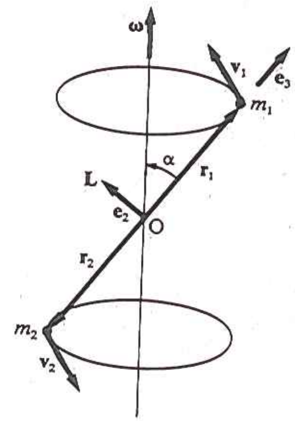


FIGURE 11-9

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha \tag{11.122}$$

Because **L** is perpendicular to the shaft and **L** rotates around **ω** as the shaft rotates, let $\mathbf{e}_2$ be along **L**:

$$L = \mathrm{L}_2\mathbf{e}_2 \tag{11.123}$$

If α is the angle between **ω** and the shaft, the components of **ω** are

$$\left.\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= \omega \sin\alpha \\ \omega_3 &= \omega\cos\alpha \end{aligned}\right\} \tag{11.124}$$

The principal axes are x_1, x_2, and x_3, and the principal moments of inertia are, from Equation 11.13a,

$$\left.\begin{aligned} I_1 &= (m_1 + m_2)b^2 \\ I_2 &= (m_1 + m_2)b^2 \\ I_3 &= 0 \end{aligned}\right\} \tag{11.125}$$

Combining Equations 11.124 and 11.125

$$\left.\begin{aligned} L_1 &= I_1\omega_1 = 0 \\ L_2 &= I_2\omega_2 = (m_1 + m_2)b^2\omega\sin\alpha \\ L_3 &= I_3\omega_3 = 0 \end{aligned}\right\} \tag{11.126}$$

which agrees with Equation 11.123.

Using Euler's equations (Equation 11.120) and $\dot{\omega} = 0$, the torque components are

$$\left.\begin{aligned} N_1 &= -(m_1 + m_2)b^2\omega^2 \sin\alpha \cos\alpha \\ N_2 &= 0 \\ N_3 &= 0 \end{aligned}\right\} \tag{11.127}$$

The torque required to maintain the motion if $\dot{\omega} = 0$ is directed along the x_1-axis.

11.9 FORCE-FREE MOTION OF A SYMMETRIC TOP

If we consider a symmetric top, that is, a rigid body with $I_1 = I_2 \neq I_3$, then the force-free Euler equations (Equation 11.114) become

$$\left.\begin{aligned} (I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_1\dot{\omega}_2 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}\right\} \tag{11.128}$$

where I_1 has been substituted for I_2. Because for force-free motion the center of mass of the body is either at rest or in uniform motion with respect to the fixed or inertial frame of reference, we can, without loss of generality, specify that the body's center of mass is at rest and located at the origin of the fixed coordinate system. We consider the case in which the angular velocity vector **ω** does not lie along a principal axis of the body, otherwise, the motion is trivial.

The first result for the motion follows from the third part of Equations 11.128, $\dot{\omega}_3 = 0$, or

$$\omega_3(t) = \text{const.} \tag{11.129}$$

The first two parts of Equation 11.128 can be written as

$$\left.\begin{aligned} \dot{\omega}_1 &= -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 \\ \dot{\omega}_2 &= \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 \end{aligned}\right\} \tag{11.130}$$

Because the terms in the parentheses are identical and composed of constants, we may define

$$\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3 \tag{11.131}$$

so that

$$\left.\begin{aligned} \dot{\omega}_1 + \Omega\omega_2 &= 0 \\ \dot{\omega}_2 - \Omega\omega_1 &= 0 \end{aligned}\right\} \tag{11.132}$$

These are coupled equations of familiar form, and we can effect a solution by multiplying the second equation by i and adding to the first:

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0 \tag{11.133}$$

If we define

$$\eta \equiv \omega_1 + i\omega_2 \tag{11.134}$$

then

$$\dot{\eta} - i\Omega\eta = 0 \tag{11.135}$$

with solution*

$$\eta(t) = Ae^{i\Omega t} \tag{11.136}$$

Thus,

$$\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t \tag{11.137}$$

and therefore

$$\left.\begin{aligned} \omega_1(t) &= A\cos\Omega t \\ \omega_2(t) &= A\sin\Omega t \end{aligned}\right\} \tag{11.138}$$

Because ω_3 = constant, we note that the magnitude of **ω** is also constant:

$$|\boldsymbol{\omega}| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = \text{constant} \tag{11.139}$$

Equations 11.138 are the parametric equations of a circle, so the projection of the vector **ω** (which is of constant magnitude) onto the x_1-x_2 plane describes a circle with time (Figure 11-10).

The x_3-axis is the symmetry axis of the body, so we find that the angular velocity vector **ω** revolves or *precesses* about the body x_3-axis with a constant angular frequency Ω. Thus, to an observer in the body coordinate system, **ω** traces out a cone around the body symmetry axis, called the **body cone**.

Because we are considering force-free motion, the angular-momentum vector **L** is stationary in the fixed coordinate system and is constant in time. An additional constant of the motion for the force-free case is the kinetic energy, or in particular, because the body's center of mass is fixed, the *rotational* kinetic energy is constant:

$$T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \text{constant} \tag{11.140}$$

But we have **L** = constant, so **ω** must move such that its projection on the stationary angular-momentum vector is constant. Thus, **ω** precesses around and makes a constant angle with the vector **L**. In such a case, **L**, **ω**, and the x_3- (body) axis (i.e., the unit vector $\mathbf{e}_3$) all lie in a *plane*. We can show this by proving that

* In general, the constant coefficient is complex, so we should properly write $A\exp(i\delta)$. For simplicity, however, we set the phase δ equal to zero; this can always be done by choosing an appropriate instant to call $t = 0$.

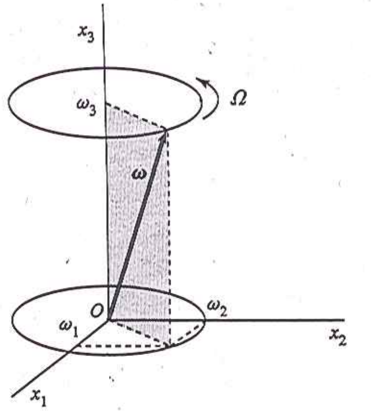


FIGURE 11-10

$\mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{e}_3) = 0$. First, $\boldsymbol{\omega} \times \mathbf{e}_3 = \omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2$. If we take the scalar product of this result with $\mathbf{L}$, we have $\mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{e}_3) = I_1\omega_1\omega_2 - I_2\omega_1\omega_2 = 0$, because $I_1 = I_2$ for the symmetric top. Therefore, if we designate the x_3'-axis in the fixed coordinate system to coincide with $\mathbf{L}$, then to an observer in the fixed system, $\boldsymbol{\omega}$ traces out a cone around the fixed x_3'-axis, called the **space cone**. The situation is then described (Figure 11-11) by one cone rolling on another, such that $\boldsymbol{\omega}$ precesses around the x_3-axis in the body system and around the x_3'-axis (or $\mathbf{L}$) in the space-fixed system.

The rate at which $\boldsymbol{\omega}$ precesses around the body symmetry axis is given by Equation 11.131:

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

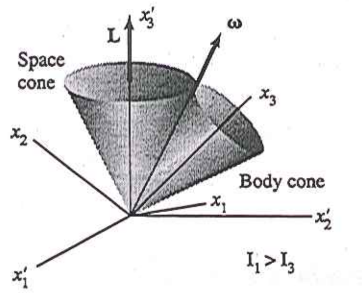


FIGURE 11-11

If $I_1 \cong I_3$, then Ω becomes very small compared with ω_3. The Earth is slightly flattened near the poles,* so its shape can be approximated by an oblate spheroid with $I_1 \cong I_3$, but with $I_3 > I_1$. If the Earth is considered to be a rigid body, then the moments I_1 and I_3 are such that $\Omega \cong \omega_3/300$. Because the period of the Earth's rotation is $2\pi/\omega = 1$ day, and because $\omega_3 \cong \omega$, the period predicted for the precession of the axis of rotation is $1/\Omega \cong 300$ days. The observed precession has an irregular period about 50% greater than that predicted on the basis of this simple theory; the deviation is ascribed to the facts that (1) the Earth is not a *rigid* body and (2) the shape is not exactly that of an oblate spheroid, but rather has a higher-order deformation and actually resembles a flattened pear.

The Earth's equatorial "bulge" together with the fact that the Earth's rotational axis is inclined at an angle of approximately 23.5° to the plane of the Earth's orbit around the sun (the **plane of the ecliptic**) produces a gravitational torque (caused by both the sun and the moon), which produces a slow precession of the Earth's axis. The period of this precessional motion is approximately 26,000 years. Thus, in different epochs, different stars become the "pole star."*

EXAMPLE 11.9

Show that the motion depicted in Figure 11-11 actually refers to the motion of a prolate object such as an elongated rod ($I_1 > I_3$), whereas for a flat disk ($I_3 > I_1$) the space cone would be inside the body cone rather than outside.

Solution: If $\mathbf{L}$ is along x_3', then the Euler angle θ (between the x_3- and x_3'-axes) is the angle between $\mathbf{L}$ and the x_3-axis. At a given instant, we align $\mathbf{e}_2$ to be in the plane defined by $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$. Then, at this same instant,

$$\left.\begin{aligned} L_1 &= 0 \\ L_2 &= L\sin\theta \\ L_3 &= L\cos\theta \end{aligned}\right\} \qquad (11.141)$$

Let α be the angle between $\boldsymbol{\omega}$ and the x_3-axis. Then, at this same instant, we have

$$\left.\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= \omega\sin\alpha \\ \omega_3 &= \omega\cos\alpha \end{aligned}\right\} \qquad (11.142)$$

* The flattening at the poles was shown by Newton to be caused by the Earth's rotation; the resulting precessional motion was first calculated by Euler.

* This precession of the equinoxes was apparently discovered by the Babylonian astronomer Cidenas in about 343 B.C.

We can also determine the components of **L** from Equation 11.34:

$$\left.\begin{aligned} L_1 &= I_1\omega_1 = 0 \\ L_2 &= I_1\omega_2 = I_1\omega \sin\alpha \\ L_3 &= I_3\omega_3 = I_3\omega \cos\alpha \end{aligned}\right\} \tag{11.143}$$

We can obtain the ratio L_2/L_3 from Equations 11.141 and 11.143,

$$\frac{L_2}{L_3} = \tan\theta = \frac{I_1}{I_3}\tan\alpha \tag{11.144}$$

so we have

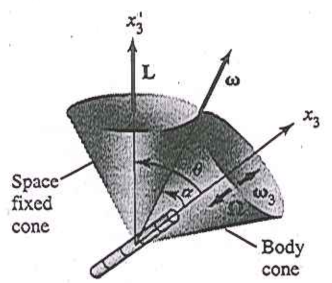
Prolate spheroid

$$I_1 > I_3, \qquad \theta > \alpha \tag{11.145a}$$

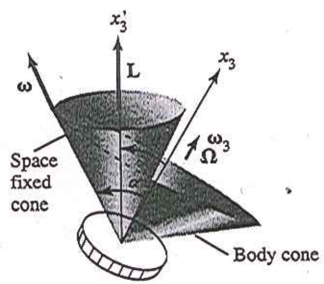
Oblate spheroid

$$I_3 > I_1, \qquad \alpha > \theta \tag{11.145b}$$

The two cases are shown in Figure 11-12. From Equation 11.131, we determine that Ω and ω_3 have the same sign if $I_3 > I_1$ but have opposite signs if $I_1 > I_3$. Thus, the sense of precession is opposite for the two cases. This fact and Equation 11.145 can be reconciled only if the space cone is outside the body cone for the prolate case but inside the body cone for the oblate case. The angular velocity **ω** defines both cones as it rotates about **L** (space cone) and the symmetry axis x_3 (body cone). The line of contact between the space and body cones is the instantaneous axis of rotation (along **ω**). At any instant, this axis is at rest, so that the body cone rolls around the space cone without slipping. In both cases, the space cone is fixed, because **L** is constant.



Prolate, $I_1 > I_3$
Ω, ω_3 have opposite signs.
(a)



Oblate, $I_3 > I_1$
Ω, ω_3 have same sign.
(b)

FIGURE 11-12

EXAMPLE 11.10

With what angular velocity does the symmetry axis (x_3) and ω rotate about the fixed angular momentum L?

Solution: Because $\mathbf{e}_3$, **ω**, and **L** are in the same plane, $\mathbf{e}_3$ and **ω** precess about **L** with the same angular velocity. In Section 11.7 we learned that $\dot{\phi}$ is the angular velocity along the x_3'-axis. If we use the same instant of time considered in the previous example (when $\mathbf{e}_2$ was in the plane of $\mathbf{e}_3$, **ω**, and **L**), then the Euler angle $\psi = 0$, and from Equation 11.102

$$\omega_2 = \dot{\phi}\sin\theta$$

and

$$\dot{\phi} = \frac{\omega_2}{\sin\theta} \tag{11.146}$$

Substituting for ω_2 from Equation 11.142, we have

$$\dot{\phi} = \frac{\omega \sin\alpha}{\sin\theta} \tag{11.147}$$

We can rewrite $\dot{\phi}$ by substituting $\sin\alpha$ from Equation 11.143 and $\sin\theta$ from Equation 11.141:

$$\dot{\phi} = \omega\frac{L_2}{I_1\omega}\frac{L}{L_2} = \frac{L}{I_1} \tag{11.148}$$

11.10 MOTION OF A SYMMETRIC TOP WITH ONE POINT FIXED

Consider a symmetric top with tip held fixed* rotating in a gravitational field. In our previous development, we have been able to separate the kinetic energy into translational and rotational parts by taking the body's center of mass to be the origin of the rotating or body coordinate system. Alternatively, if we can choose the origins of the fixed and the body coordinate systems to coincide, then the translational kinetic energy vanishes, because $\mathbf{V} = \dot{\mathbf{R}} = 0$. Such a choice is quite convenient for discussing the top, because the stationary tip may then be taken as the origin for both coordinate systems. Figure 11-13 shows the Euler angles for this situation. The x_3'- (fixed) axis corresponds to the vertical, and we choose the x_3- (body) axis to be the symmetry axis of the top. The distance from the fixed tip to the center of mass is h, and the mass of the top is M.

* This problem was first solved in detail by Lagrange in *Mécanique analytique.*

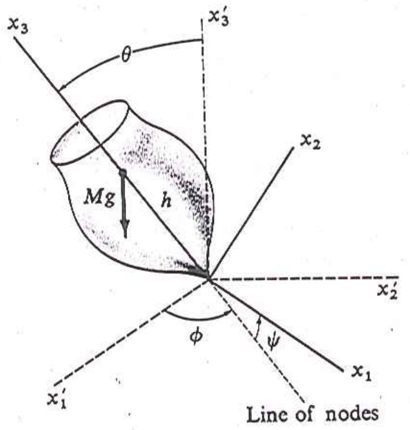


FIGURE 11-13

Because we have a symmetric top, the principal moments of inertia about the x_1- and x_2-axes are equal: $I_1 = I_2$. We assume $I_3 \neq I_1$. The kinetic energy is then given by

$$T = \frac{1}{2}\sum_i I_i\omega_i^2 = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 \tag{11.149}$$

According to Equation 11.102, we have

$$\begin{aligned}\omega_1^2 &= (\dot\phi \sin\theta \sin\psi + \dot\theta\cos\psi)^2\\ &= \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta\sin\psi\cos\psi + \dot\theta^2\cos^2\psi\\ \omega_2^2 &= (\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)^2\\ &= \dot\phi^2\sin^2\theta\cos^2\psi - 2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\sin^2\psi\end{aligned}$$

so that

$$\omega_1^2 + \omega_2^2 = \dot\phi^2\sin^2\theta + \dot\theta^2 \tag{11.150a}$$

and

$$\omega_3^2 = (\dot\phi\cos\theta + \dot\psi)^2 \tag{11.150b}$$

Therefore,

$$T = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 \tag{11.151}$$

Because the potential energy is $Mgh\cos\theta$, the Lagrangian becomes

$$L = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - Mgh\cos\theta \tag{11.152}$$

The Lagrangian is cyclic in both the ϕ- and ψ-coordinates. The momenta conjugate to these coordinates are therefore constants of the motion:

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta = \text{constant} \tag{11.153}$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = \text{constant} \tag{11.154}$$

Because the cyclic coordinates are *angles*, the conjugate momenta are *angular momenta*—the angular momenta along the axes for which ϕ and ψ are the rotation angles, that is, the x_3'- (or vertical) axis and the x_3- (or body symmetry) axis, respectively. We note that this result is ensured by the construction shown in Figure 11-13, because the gravitational torque is directed along the line of nodes. Hence, the torque can have no component along either the x_3'- or the x_3-axis, both of which are perpendicular to the line of nodes. Thus, the angular momenta along these axes are constants of the motion.

Equations 11.153 and 11.154 can be solved for $\dot\phi$ and $\dot\psi$ in terms of θ. From Equation 11.154, we can write

$$\dot\psi = \frac{p_\psi - I_3\dot\phi\cos\theta}{I_3} \tag{11.155}$$

and substituting this result into Equation 11.153, we find

$$(I_1\sin^2\theta + I_3\cos^2\theta)\dot\phi + (p_\psi - I_3\dot\phi\cos\theta)\cos\theta = p_\phi$$

or

$$(I_1\sin^2\theta)\dot\phi + p_\psi\cos\theta = p_\phi$$

so that

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \tag{11.156}$$

Using this expression for $\dot\phi$ in Equation 11.155, we have

$$\dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta} \tag{11.157}$$

By hypothesis, the system we are considering is conservative; we therefore have the further property that the total energy is a constant of the motion:

$$E = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta = \text{constant} \tag{11.158}$$

Using the expression for ω_3 (e.g., see Equation 11.102), we note that Equation 11.154 can be written as

$$p_\psi = I_3\omega_3 = \text{constant} \tag{11.159a}$$

or

$$I_3\omega_3^2 = \frac{p_\psi^2}{I_3} = \text{constant} \tag{11.159b}$$

Therefore, not only is E a constant of the motion, but so is $E - \frac{1}{2}I_3\omega_3^2$; we let this quantity be E':

$$E' \equiv E - \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + Mgh\cos\theta = \text{constant} \tag{11.160}$$

Substituting into this equation the expression for $\dot\phi$ (Equation 11.156), we have

$$E' = \frac{1}{2}I_1\dot\theta^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta \tag{11.161}$$

which we can write as

$$E' = \frac{1}{2}I_1\dot\theta^2 + V(\theta) \tag{11.162}$$

where $V(\theta)$ is an "effective potential" given by

$$V(\theta) \equiv \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta \tag{11.163}$$

Equation 11.162 can be solved to yield $t(\theta)$:

$$t(\theta) = \int \frac{d\theta}{\sqrt{(2/I_1)[E' - V(\theta)]}} \tag{11.164}$$

This integral can (formally, at least) be inverted to obtain $\theta(t)$, which, in turn, can be substituted into Equations 11.156 and 11.157 to yield $\phi(t)$ and $\psi(t)$. Because the Euler angles θ, ϕ, ψ completely specify the orientation of the top, the results for $\theta(t)$, $\phi(t)$, and $\psi(t)$ constitute a complete solution for the problem. It should be clear that such a procedure is complicated and not very illuminating. But we can obtain some qualitative features of the motion by examining the preceding equations in a manner analogous to that used for treating the motion of a particle in a central-force field (see Section 8.6).

Figure 11-14 shows the form of the effective potential $V(\theta)$ in the range $0 \le \theta \le \pi$, which clearly is the physically limited region for θ. This energy diagram indicates that for any general values of E' (e.g., the value represented by E_1') the motion is limited by two extreme values of θ—that is, θ_1 and θ_2, which correspond to the turning points of the central-force problem and are roots of the denominator in Equation 11.164. Thus we find that the inclination of the rotating top is, in general, confined to the region $\theta_1 \le \theta \le \theta_2$. For the case that $E' = E_2' = V_{min}$, θ is limited to the single value θ_0, and the motion is a steady precession at a fixed angle of inclination. Such motion is similar to the occurrence of circular orbits in the central-force problem.

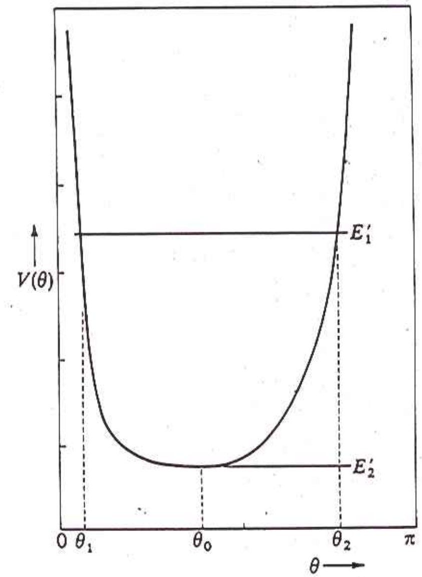


FIGURE 11-14

The value of θ_0 can be obtained by setting the derivative of $V(\theta)$ equal to zero. Thus,

$$\left.\frac{\partial V}{\partial\theta}\right|_{\theta=\theta_0} = \frac{-\cos\theta_0(p_\phi - p_\psi\cos\theta_0)^2 + p_\psi\sin^2\theta_0(p_\phi - p_\psi\cos\theta_0)}{I_1\sin^3\theta_0} - Mgh\sin\theta_0 = 0 \tag{11.165}$$

If we define

$$\beta \equiv p_\phi - p_\psi\cos\theta_0 \tag{11.166}$$

then Equation 11.165 becomes

$$(\cos\theta_0)\beta^2 - (p_\psi\sin^2\theta_0)\beta + (MghI_1\sin^4\theta_0) = 0 \tag{11.167}$$

This is a quadratic in β and can be solved with the result

$$\beta = \frac{p_\psi\sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right) \tag{11.168}$$

Because β must be a real quantity, the radicand in Equation 11.168 must be positive. If $\theta_0 < \pi/2$, we have

$$p_\psi^2 \ge 4MghI_1\cos\theta_0 \tag{11.169}$$

But from Equation 11.159a, $p_\psi = I_3\omega_3$; thus,

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1 \cos\theta_0} \qquad \textbf{(11.170)}$$

We therefore conclude that a steady precession can occur at the fixed angle of inclination θ_0 only if the angular velocity of spin is larger than the limiting value given by Equation 11.170.

From Equation 11.156, we note that we can write (for $\theta = \theta_0$)

$$\dot\phi_0 = \frac{\beta}{I_1 \sin^2\theta_0} \qquad \textbf{(11.171)}$$

We therefore have two possible values of the precessional angular velocity $\dot\phi_0$, one for each of the values of β given by Equation 11.168:

$$\dot\phi_{0(+)} \rightarrow \text{Fast precession}$$

and

$$\dot\phi_{0(-)} \rightarrow \text{Slow precession}$$

If ω_3 (or p_ψ) is large (a fast top), then the second term in the radicand of Equation 11.168 is small, and we may expand the radical. Retaining only the first nonvanishing term in each case, we find

$$\left.\begin{aligned} \dot\phi_{0(+)} &\cong \frac{I_3\omega_3}{I_1 \cos\theta_0} \\ \dot\phi_{0(-)} &\cong \frac{Mgh}{I_3\omega_3} \end{aligned}\right\} \qquad \textbf{(11.172)}$$

It is the slower of the two possible precessional angular velocities, $\dot\phi_{0(-)}$, that is usually observed.

The preceding results apply if $\theta_0 < \pi/2$; but if* $\theta_0 > \pi/2$, the radicand in Equation 11.168 is always positive and there is no limiting condition on ω_3. Because the radical is greater than unity in such a case, the values of $\dot\phi_0$ for fast and slow precession have opposite signs; that is, for $\theta_0 > \pi/2$, the fast precession is in the same direction as that for $\theta_0 < \pi/2$, but the slow precession takes place in the opposite sense.

For the general case, in which $\theta_1 < \theta < \theta_2$, Equation 11.156 indicates that $\dot\phi$ may or may not change sign as θ varies between its limits—depending on the values of p_ϕ and p_ψ. If $\dot\phi$ does not change sign, the top precesses monotonically around the x_3'-axis (see Figure 11-13), and the x_3- (or symmetry) axis oscillates between $\theta = \theta_1$ and $\theta = \theta_2$. This phenomenon is called **nutation**; the path

* If $\theta_0 > \pi/2$, the fixed tip of the top is at a position *above* the center of mass. Such motion is possible, for example, with a gyroscopic top whose tip is actually a ball and rests in a cup that is fixed atop a pedestal.

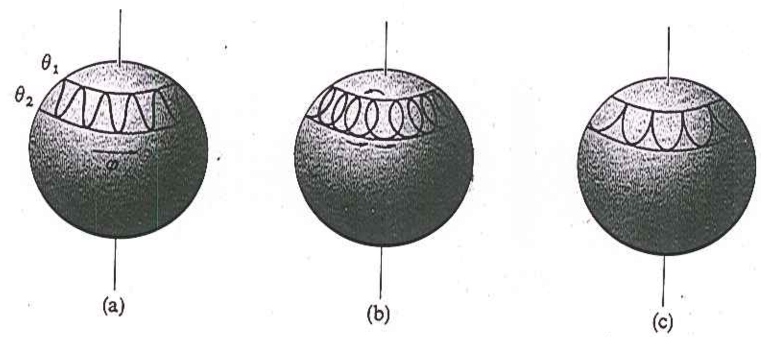


FIGURE 11-15

described by the projection of the body symmetry axis on a unit sphere in the fixed system is shown in Figure 11-15a.

If $\dot\phi$ does change sign between the limiting values of θ, the precessional angular velocity must have opposite signs at $\theta = \theta_1$ and $\theta = \theta_2$. Thus, the nutational-precessional motion produces the looping motion of the symmetry axis depicted in Figure 11-15b.

Finally, if the values of p_ϕ and p_ψ are such that

$$(p_\phi - p_\psi \cos\theta)|_{\theta=\theta_1} = 0, \qquad \textbf{(11.173)}$$

then

$$\dot\phi|_{\theta=\theta_1} = 0, \qquad \dot\theta|_{\theta=\theta_1} = 0 \qquad \textbf{(11.174)}$$

Figure 11-15c shows the resulting cusplike motion. It is just this case that corresponds to the usual method of starting a top. First, the top is spun around its axis, then it is given a certain initial tilt and released. Thus, initial conditions are $\theta = \theta_1$ and $\dot\theta = 0 = \dot\phi$. Because the first motion of the top is to begin to fall in the gravitational field, the conditions are exactly those of Figure 11-15c, and the cusplike motion ensues. Figures 11-15a and 11-15b correspond to the motion in the event that there is an initial angular velocity $\dot\phi$ either in the direction of or opposite to the direction of precession.

11.11 STABILITY OF RIGID-BODY ROTATIONS

We now consider a rigid body undergoing force-free rotation around one of its principal axes and inquire whether such motion is stable. "Stability" here means, as before (see Section 8.10), that if a small perturbation is applied to the system, the motion will either return to its former mode or will perform small oscillations about it.

We choose for our discussion a general rigid body for which all the principal moments of inertia are distinct, and we label them such that $I_3 > I_2 > I_1$. We let the body axes coincide with the principal axes, and we start with the body rotating around the x_1-axis—that is, around the principal axis associated with the moment of inertia I_1. Then,

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 \tag{11.175}$$

If we apply a small perturbation, the angular velocity vector assumes the form

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 \tag{11.176}$$

where λ and μ are small quantities and correspond to the parameters used previously in other perturbation expansions. (λ and μ are sufficiently small so that their product can be neglected compared with all other quantities of interest to the discussion.)

The Euler equations (see Equation 11.114) become

$$\left.\begin{aligned} (I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} &= 0 \\ (I_1 - I_2)\lambda\omega_1 - I_3\dot{\mu} &= 0 \end{aligned}\right\} \tag{11.177}$$

Because $\lambda\mu \approx 0$, the first of these equations requires $\dot{\omega}_1 = 0$, or $\omega_1 =$ constant. Solving the other two equations for $\dot{\lambda}$ and $\dot{\mu}$, we find

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu \tag{11.178}$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\lambda \tag{11.179}$$

where the terms in parentheses are both constants. These are coupled equations, but they cannot be solved by the method used in Section 11.9, because the constants in the two equations are different. The solution can be obtained by first differentiating the equation for $\dot{\lambda}$:

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\dot{\mu} \tag{11.180}$$

The expression for $\dot{\mu}$ can now be substituted in this equation:

$$\ddot{\lambda} + \left(\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2\right)\lambda = 0 \tag{11.181}$$

The solution to this equation is

$$\lambda(t) = Ae^{i\Omega_{1\lambda}t} + Be^{-i\Omega_{1\lambda}t} \tag{11.182}$$

where

$$\Omega_{1\lambda} \equiv \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{11.183}$$

and where the subscripts 1 and λ indicate that we are considering the solution for λ when the rotation is around the x_1-axis.

By hypothesis, $I_1 < I_3$ and $I_1 < I_2$, so $\Omega_{1\lambda}$ is real. The solution for $\lambda(t)$ therefore represents oscillatory motion with a frequency $\Omega_{1\lambda}$. We can similarly investigate $\mu(t)$, with the result that $\Omega_{1\mu} = \Omega_{1\lambda} \equiv \Omega_1$. Thus, the small perturbations introduced by forcing small x_2- and x_3-components on $\boldsymbol{\omega}$ do not increase with time but oscillate around the equilibrium values $\lambda = 0$ and $\mu = 0$. Consequently, the rotation around the x_1-axis is stable.

If we consider rotations around the x_2- and x_3-axes, we can obtain expressions for Ω_2 and Ω_3 from Equation 11.183 by permutation:

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{11.184a}$$

$$\Omega_2 = \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \tag{11.184b}$$

$$\Omega_3 = \omega_3\sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}} \tag{11.184c}$$

But because $I_1 < I_2 < I_3$, we have

$$\Omega_1, \Omega_3 \text{ real}, \qquad \Omega_2 \text{ imaginary}$$

Thus, when the rotation takes place around either the x_1- or x_3-axes, the perturbation produces oscillatory motion and the rotation is stable. When the rotation takes place around x_2, however, the fact that Ω_2 is imaginary results in the perturbation increasing with time without limit; such motion is unstable.

Because we have assumed a completely arbitrary rigid body for this discussion, we conclude that rotation around the principal axis corresponding to either the greatest or smallest moment of inertia is stable and that rotation around the principal axis corresponding to the intermediate moment is unstable. We can demonstrate this effect with, say, a book (kept closed by tape or a rubber band). If we toss the book into the air with an angular velocity around one of the principal axes, the motion is unstable for rotation around the intermediate axis and stable for the other two axes.

If two of the moments of inertia are equal ($I_1 = I_2$, say), then the coefficient of λ in Equation 11.179 vanishes, and we have $\dot{\mu} = 0$ or $\mu(t) =$ constant. Equation 11.178 for λ can therefore be integrated to yield

$$\lambda(t) = C + Dt \tag{11.185}$$

and the perturbation increases linearly with the time; the motion around the x_1-axis is therefore unstable. We find a similar result for motion around the x_2-axis. Stability exists only for the x_3-axis, independent of whether I_3 is greater or less than $I_1 = I_2$.

A good example of the stability of rotating objects is seen by the satellites put into space by the space shuttle orbiter. When the satellites are ejected from the payload bay, they are normally spinning in a stable configuration. In May 1992, when the astronauts attempted to grab in space the Intelsat satellite to attach a rocket that would insert it into geosynchronous orbit, the spinning satellite was slowed down and stopped before the astronaut attempted to attach a grappling fixture to bring it into the payload bay. After each futile attempt, when the grappling fixture failed, the satellite tumbled even more. After each of two unsuccessful days of trying to attach the grappling fixture, the astronauts had to abort their attempts because of the increased tumbling. Ground controllers would then require a few hours to restabilize the satellite using jet thrusters. The satellite would be left in a stable configuration of spinning slowly about its cyclindrical symmetry axis (a principal axis). Finally, on the third day, three astronauts went outside the orbiter, grabbed the slightly rotating satellite, stopped it, and put it into the payload bay where the rocket skirt was attached. The Intelsat satellite was finally successfully placed into orbit in time to broadcast the 1992 Barcelona Olympic summer games.

PROBLEMS

11-1. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous sphere of radius R and mass M. (Choose the origin at the center of the sphere.)

11-2. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous cone of mass M whose height is h and whose base has a radius R. Choose the x_3-axis along the axis of symmetry of the cone. Choose the origin at the apex of the cone, and calculate the elements of the inertia tensor. Then make a transformation such that the center of mass of the cone becomes the origin, and find the principal moments of inertia.

11-3. Calculate the moments of inertia I_1, I_2, and I_3 for a homogeneous ellipsoid of mass M with axes' lengths $2a > 2b > 2c$.

11-4. Consider a thin rod of length l and mass m pivoted about one end. Calculate the moment of inertia. Find the point at which, if all the mass were concentrated, the moment of inertia about the pivot axis would be the same as the real moment of inertia. The distance from this point to the pivot is called the **radius of gyration.**

11-5. (a) Find the height at which a billiard ball should be struck so that it will roll with no initial slipping. (b) Calculate the optimum height of the rail of a billiard table. On what basis is the calculation predicated?

11-6. Two spheres are of the same diameter and same mass, but one is solid and the other is a hollow shell. Describe in detail a nondestructive experiment to determine which is solid and which is hollow.

11-7. A homogeneous disk of radius R and mass M rolls without slipping on a horizontal surface and is attracted to a point a distance d below the plane. If the force of attraction is proportional to the distance from the disk's center of mass to the force center, find the frequency of oscillations around the position of equilibrium.

11-8. A door is constructed of a thin homogeneous slab of material; it has a width of 1 m. If the door is opened through 90°, it is found that on release it closes itself in 2 s. Assume that the hinges are frictionless, and show that the line of hinges must make an angle of approximately 3° with the vertical.

11-9. A homogeneous slab of thickness a is placed atop a fixed cylinder of radius R whose axis is horizontal. Show that the condition for stable equilibrium of the slab, assuming no slipping, is $R > a/2$. What is the frequency of small oscillations? Sketch the potential energy U as a function of the angular displacement θ. Show that there is a minimum at $\theta = 0$ for $R > a/2$ but not for $R < a/2$.

11-10. A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A particle of mass m, initially at one pole, moves with a constant velocity v along a great circle of the sphere. Show that, when the particle has reached the other pole, the rotation of the sphere will have been retarded by an angle

$$\alpha = \omega T\left(1 - \sqrt{\frac{2M}{2M + 5m}}\right)$$

where T is the total time required for the particle to move from one pole to the other.

11-11. A homogeneous cube, each edge of which has a length l, is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a small displacement and allowed to fall. Show that the angular velocity of the cube when one face strikes the plane is given by

$$\omega^2 = A\frac{g}{l}(\sqrt{2} - 1)$$

where $A = 3/2$ if the edge cannot slide on the plane and where $A = 12/5$ if sliding can occur without friction.

11-12. Show that none of the principal moments of inertia can exceed the sum of the other two.

11-13. A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$m_1 = 3m, \quad (b, 0, b)$$

$$m_2 = 4m, \quad (b, b, -b)$$

$$m_3 = 2m, \quad (-b, b, 0)$$

Find the inertia tensor, principal axes, and principal moments of inertia.

11-14. Determine the principal axes and principal moments of inertia of a uniformly solid hemisphere of radius b and mass m about its center of mass.

11-15. If a physical pendulum has the same period of oscillation when pivoted about either of two points of unequal distances from the center of mass, show that the length of the simple pendulum with the same period is equal to the separation of the pivot points. Such a physical pendulum, called **Kater's reversible pendulum,** at one time provided the most accurate way (to about 1 part in 10^5) to measure the acceleration of gravity.* Discuss the advantages of Kater's pendulum over a simple pendulum for such a purpose.

* First used in 1818 by Captain Henry Kater (1777–1835), but the method was apparently suggested somewhat earlier by Bohnenberger. The theory of Kater's pendulum was treated in detail by Friedrich Wilhelm Bessel (1784–1846) in 1826.

11-16. Consider the following inertia tensor:

$$\{\mathbf{I}\} = \begin{Bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{Bmatrix}$$

Perform a rotation of the coordinate system by an angle θ about the x_3-axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements A, B, and C.

11-17. Consider a thin homogeneous plate that lies in the x_1–x_2 plane. Show that the inertia tensor takes the form

$$\{\mathbf{I}\} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$

11-18. If, in the previous problem, the coordinate axes are rotated through an angle θ about the x_3-axis, show that the new inertia tensor is

$$\{\mathbf{I}\} = \begin{Bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{Bmatrix}$$

where

$$A' = A\cos^2\theta - C\sin 2\theta + B\sin^2\theta$$

$$B' = A\sin^2\theta + C\sin 2\theta + B\cos^2\theta$$

$$C' = C\cos 2\theta - \tfrac{1}{2}(B-A)\sin 2\theta$$

and hence show that the x_1- and x_2-axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C}{B-A}\right)$$

11-19. Consider a plane homogeneous plate of density ρ bounded by the logarithmic spiral $r = ke^{\alpha\theta}$ and the radii $\theta = 0$ and $\theta = \pi$. Obtain the inertia tensor for the origin at $r = 0$ if the plate lies in the x_1–x_2 plane. Perform a rotation of the coordinate axes to obtain the principal moments of inertia, and use the results of the previous problem to show that they are

$$I_1' = \rho k^4 P(Q-R), \qquad I_2' = \rho k^4 P(Q+R), \qquad I_3' = I_1' + I_2'$$

where

$$P = \frac{e^{4\pi\alpha}-1}{16(1+4\alpha^2)}, \qquad Q = \frac{1+4\alpha^2}{2\alpha}, \qquad R = \sqrt{1+4\alpha^2}$$

11-20. A uniform rod of strength b stands vertically upright on a rough floor and then tips over. What is the rod's angular velocity when it hits the floor?

11-21. The proof represented by Equations 11.54–11.61 is expressed entirely in the summation convention. Rewrite this proof in matrix notation.

11-22. The **trace** of a tensor is defined as the sum of the diagonal elements:

$$\text{tr}\{\mathbf{I}\} \equiv \sum_k I_{kk}$$

Show, by performing a similarity tranformation, that the trace is an invariant quantity. In other words, show that

$$\text{tr}\{\mathbf{I}\} = \text{tr}\{\mathbf{I}'\}$$

where $\{\mathbf{I}\}$ is the tensor in one coordinate system and $\{\mathbf{I}\}'$ is the tensor in a coordinate system rotated with respect to the first system. Verify this result for the different forms of the inertia tensor for a cube given in several examples in the text.

11-23. Show by the method used in the previous problem that the *determinant* of the elements of a tensor is an invariant quantity under a similarity transformation. Verify this result also for the case of the cube.

11-24. Find the frequency of small oscillations for a thin homogeneous plate if the motion takes place in the plane of the plate and if the plate has the shape of an equilateral triangle and is suspended **(a)** from the midpoint of one side and **(b)** from one apex.

11-25. Consider a thin disk composed of two homogeneous halves connected along a diameter of the disk. If one half has density ρ and the other has density 2ρ, find the expression for the Lagrangian when the disk rolls without slipping along a horizontal surface. (The rotation takes place in the plane of the disk.)

11-26. Obtain the components of the angular velocity vector $\boldsymbol{\omega}$ (see Equation 11.102) directly from the transformation matrix $\boldsymbol{\lambda}$ (Equation 11.99).

11-27. A symmetric body moves without the influence of forces or torques. Let x_3 be the symmetry axis of the body and $\mathbf{L}$ be along x_3'. The angle between $\boldsymbol{\omega}$ and x_3 is α. Let $\boldsymbol{\omega}$ and $\mathbf{L}$ initially be in the x_2–x_3 plane. What is the angular velocity of the symmetry axis about $\mathbf{L}$ in terms of I_1, I_3, ω, and α?

11-28. Show from Figure 11-7c that the components of $\boldsymbol{\omega}$ along the fixed (x_i') axes are

$$\omega_1' = \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi$$

$$\omega_2' = \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi$$

$$\omega_3' = \dot\psi\cos\theta + \dot\phi$$

11-29. Investigate the motion of the symmetric top discussed in Section 11.10 for the case in which the axis of rotation is vertical (i.e., the x_3'- and x_3-axes coincide). Show that the motion is either stable or unstable depending on whether the quantity $4I_1 Mhg/I_3^2\omega_3^2$ is less than or greater than unity. Sketch the effective potential $V(\theta)$ for the two cases, and point out the features of these curves that determine whether the motion is stable. If the top is set spinning in the stable configuration, what is the effect as friction gradually reduces the value of ω_3? (This is the case of the "sleeping top.")

11-30. Refer to the discussion of the symmetric top in Section 11.10. Investigate the equation for the turning points of the nutational motion by setting $\dot\theta = 0$ in Equation 11.162. Show that the resulting equation is a cubic in $\cos\theta$ and has two real roots and one imaginary root for θ.

11-31. Consider a thin homogeneous plate with principal momenta of inertia

$$I_1 \quad \text{along the principal axis } x_1$$

$$I_2 > I_1 \quad \text{along the principal axis } x_2$$

$$I_3 = I_1 + I_2 \quad \text{along the principal axis } x_3$$

Let the origins of the x_i and x'_i systems coincide and be located at the center of mass O of the plate. At time $t = 0$, the plate is set rotating in a force-free manner with an angular velocity Ω about an axis inclined at an angle α from the plane of the plate and perpendicular to the x_2-axis. If $I_1/I_2 \equiv \cos 2\alpha$, show that at time t the angular velocity about the x_2-axis is

$$\omega_2(t) = \Omega \cos \alpha \tanh(\Omega t \sin \alpha)$$

C H A P T E R

12

COUPLED OSCILLATIONS

12.1 INTRODUCTION

In Chapter 3, we examined the motion of an oscillator subjected to an external driving force. The discussion was limited to the case in which the driving force is periodic; that is, the driver is itself a harmonic oscillator. We considered the action of the driver on the oscillator, but we did not include the feedback effect of the oscillator on the driver. In many instances, ignoring this effect is unimportant, but if two (or many) oscillators are connected in such a way that energy can be transferred back and forth between (or among) them, the situation becomes the more complicated case of **coupled oscillations.*** Motion of this type can be quite complex (the motion may not even be periodic), but we can always describe the motion of any oscillatory system in terms of **normal coordinates**, which have the property that each oscillates with a single, well-defined frequency; that is, the normal coordinates are constructed in such a way that no coupling occurs among them, even though there is coupling among the ordinary (rectangular) coordinates describing the positions of particles. Initial conditions can always be prescribed for the system so that in the subsequent motion only one normal coordinate varies with time. In this circumstance, we say that one of the **normal modes** of the system has been

*The general theory of the oscillatory motion of a system of particles with a finite number of degrees of freedom was formulated by Lagrange during the period 1762–1765, but the pioneering work had been done in 1753 by Daniel Bernoulli (1700–1782).