Solutions

Phys 501: Midterm Exam 2

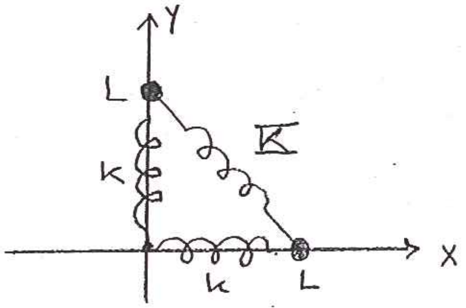
Fall 2013

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours.

Problem 1 (30 points) Two particles of equal mass m are attached via identical springs of spring constant k to the origin of a Cartesian coordinate system. One of the particles are constrained to move on the x-axis and the other on the y-axis. The particles are also connected to one another by another spring of spring constant K. The system is at equilibrium when the two particles are respectively located at the points $(L, 0, 0)$ and $(0, L, 0)$ with L being the equilibrium length of the identical springs (See the figure below.) Find the normal modes of this system and the corresponding solutions of equations of motion.
Hint: Use $q^1 := x - L$ and $q^2 := y - L$ as the generalized coordinates.



Problem 2 (25 points) A Hamiltonian system with a single degree of freedom is determined by the Hamiltonian $H(q,p) = \dfrac{e^{-q/a}p^2}{2m}$, where a and m are positive real parameters. Find and solve Hamilton's equations to determine $q(t)$ and $p(t)$ for $t > 0$ provided that $q(0) = 0$ and $p(0) = p_0$ for some $p_0 \in \mathbb{R}$.

Problem 3 (15 points) Consider the system described in Problem 2. Find the time-independent Hamilton-Jacobi equation for this system and use Hamilton-Jacobi formulation to determine $q(t)$ and $p(t)$ for $t > 0$ such that $q(0) = 0$ and $p(0) = p_0$ for some $p_0 \in \mathbb{R}$.

Problem 4 Consider a Hamiltonian system with a single real degree of freedom and a time-dependent Hamiltonian $H = H(q,p,t)$ where $(p,q) \in \mathbb{R}^2$. Suppose that we perform a Type 3 canonical transformation, i.e., a canonical transformation generated by $F_3(p, \tilde{q}, t)$, where $(\tilde{q}, \tilde{p})$ denote the transformed dynamical variables.

4.a (10 points) Express q, $\tilde{p}$, and the transformed Hamiltonian K in terms of F_3 and H.

4.b (10 points) Find the analogue of the Hamilton-Jacobi equation that is satisfied by F_3 by demanding that $K = 0$.

4.c (10 points) Derive the analogue of time-independent Hamilton-Jacobi equation for this canonical transformation for $H = \dfrac{p^2}{2m} + \lambda\, q^4$, where λ is a positive real parameter.

Problem 1 : $L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{k}{2}\left[(x-L)^2+(y-L)^2\right] +$

$$- \frac{\bar{K}}{2}\left[\sqrt{x^2+y^2} - \sqrt{2}L\right]^2$$

$q^1 = x - L, \quad q^2 = y - L, \quad \dot{x} = \dot{q}^1, \quad \dot{y} = \dot{q}^2$

$$\sqrt{x^2+y^2} = \left[(q^1+L)^2 + (q^2+L)^2\right]^{\frac12} = L\left[\left(1+\frac{q^1}{L}\right)^2 + \left(1+\frac{q^2}{L}\right)^2\right]^{1/2}$$

$$= L\left(1+\frac{2q^1}{L}+1+\frac{2q^2}{L}\right)^{1/2} + O\left(\frac{q^i}{L}\right)^2$$

$$= \sqrt{2}L\left(1+\frac{q^1+q^2}{L}\right)^{1/2} + O\left(\frac{q^i}{L}\right)^2$$

$$= \sqrt{2}L\left(1+\frac{q^1+q^2}{2L}\right) + O\left(\frac{q^i}{L}\right)^2$$

$$= \sqrt{2}L + \frac{q^1+q^2}{\sqrt{2}} + O\left(\frac{q^i}{L}\right)^2$$

For $\frac{|q^i|}{L} \ll 1$, we therefore have.

$$L \approx \frac{m}{2}\left[(\dot{q}^1)^2 + (\dot{q}^2)^2\right] - \frac{k}{2}\left[(q^1)^2+(q^2)^2\right] - \frac{\bar{K}}{2}\left\{\frac{(q^1+q^2)^2}{2}\right\}$$

$$\approx \frac{m}{2}\left[(\dot{q}^1)^2 + (\dot{q}^2)^2\right] - \left(\frac{2k+\bar{K}}{4}\right)\left[(q^1)^2+(q^2)^2\right] - \frac{\bar{K}}{2}q^1q^2$$

$$\approx \frac12 M_{ij}\dot{q}^i\dot{q}^j - \frac12 K_{ij}q^iq^j \quad \text{where}$$

$$\mathbb{M} = [M_{ij}] = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \mathbb{K} = [K_{ij}] = \begin{bmatrix}\frac{2k+\bar{K}}{2} & \frac{\bar{K}}{2}\\ \frac{\bar{K}}{2} & \frac{2k+\bar{K}}{2}\end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \Rightarrow M_{ij}\ddot{q}^j + K_{ij}q^j = 0$$

$$\Rightarrow \boxed{\ddot{q}^k + (\mathbb{M}^{-1}\mathbb{K})_{kj}q^j = 0}$$

$$\Omega^2 := \mathbb{M}^{-1}\mathbb{K} = \frac{1}{m}\mathbb{K} = \frac{1}{2m}\begin{bmatrix}2k+\bar{K} & \bar{K}\\ \bar{K} & 2k+\bar{K}\end{bmatrix}$$

Normal modes are given by the eigenvectors of Ω^2.

$$\left[\Omega^2 - \omega^2 \mathbb{I}\right] \vec{a} = \vec{0} \qquad \Rightarrow$$

$$\frac{1}{2m}\begin{bmatrix} 2\kappa + \bar{K} - 2m\omega^2 & \bar{K} \\ \bar{K} & 2\kappa + \bar{K} - 2m\omega^2 \end{bmatrix} \vec{a} = \vec{0}$$

$$\Rightarrow \det \begin{bmatrix} 2\kappa + \bar{K} - 2m\omega^2 & \bar{K} \\ \bar{K} & 2\kappa + \bar{K} - 2m\omega^2 \end{bmatrix} = 0$$

$$\Rightarrow \quad (2\kappa + \bar{K} - 2m\omega^2)^2 - \bar{K}^2 = 0$$

$$\Rightarrow \quad 2m\omega^2 = 2\kappa + \bar{K} \pm \bar{K} = \begin{cases} 2(\kappa + \bar{K}) \\ 2\kappa \end{cases}$$

$$\Rightarrow \quad \omega = \left\{ \begin{aligned} \omega_+ &:= \sqrt{\frac{\kappa + \bar{K}}{m}} \\ \omega_- &:= \sqrt{\frac{\kappa}{m}} \end{aligned} \right\} \text{ Normal mode frequencies}$$

<u>For $\omega = \omega_+$</u>, $\Omega^2 - \omega^2 \mathbb{I} = \frac{1}{2m}\begin{bmatrix} -\bar{K} & \bar{K} \\ \bar{K} & -\bar{K} \end{bmatrix} = \frac{\bar{K}}{2m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{a} = \vec{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a^1 \\ a^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -a^1 + a^2 = 0$$

$$a^2 = a^1 =: C_+$$

$$\Rightarrow \vec{a} = \vec{a}_+ := C_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_+ \in \mathbb{R} \setminus \{0\}$$

<u>For $\omega = \omega_-$</u>, $\Omega^2 - \omega^2 \mathbb{I} = \frac{1}{2m}\begin{bmatrix} \bar{K} & \bar{K} \\ \bar{K} & \bar{K} \end{bmatrix} = \frac{\bar{K}}{2m}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{a} = \vec{0} \Rightarrow a^1 + a^2 = 0 \Rightarrow a^1 = -a^2 =: C_1$$

$$\vec{a} = \vec{a}_- := C_+ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C_- \in \mathbb{R} \setminus \{0\}$$

So normal modes correspond to complex solutions

$$\vec{q}_\pm(t) = \begin{bmatrix} q^1_\pm(t) \\ q^2_\pm(t) \end{bmatrix} = e^{i\omega_\pm t} \vec{a}_\pm$$

which are equivalent to the real solutions

$$\begin{bmatrix} q^1_\pm(t) \\ q^2_\pm(t) \end{bmatrix} = \cos(\omega_\pm t + \varphi_\pm)\, \vec{a}_\pm$$

Problem 2 : $H = \dfrac{e^{-\frac{q}{a}} p^2}{2m}$, $q(0) = 0$, $p(0) = P_0$

$$\dot{q} = \frac{\partial H}{\partial p} = \left(\frac{e^{-\frac{q}{a}}}{m}\right) p \qquad (1)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = + \frac{e^{-\frac{q}{a}} p^2}{2ma} \qquad (2)$$

$$\Rightarrow \frac{dp}{dq} = \frac{p^2}{2ap} = \frac{p}{2a} \Rightarrow \frac{dp}{p} = \frac{dq}{2a}$$

$$\Rightarrow d \ln p = d\left(\frac{q}{2a}\right) \Rightarrow p = c\, e^{\frac{q}{2a}}$$

For $t = 0$, $c = p(0) = P_0 \;\Rightarrow\; \boxed{p = P_0 e^{\frac{q}{2a}}}$ (3)

$$(1) \,\&\, (3) \Rightarrow \dot{q} = \left(\frac{e^{-\frac{q}{a}}}{m}\right)\left(P_0 e^{\frac{q}{2a}}\right) = \frac{P_0 e^{-\frac{q}{2a}}}{m}$$

$$\Rightarrow e^{\frac{q}{2a}} dq = \frac{P_0}{m} dt$$

$$\Rightarrow 2a\, e^{\frac{q}{2a}} = \frac{P_0 t}{m} + k \Rightarrow 2a\left(e^{\frac{q}{2a}} - 1\right) = \frac{P_0 t}{m}$$

at $t = 0$, $q = 0$, $\Rightarrow 2a = k$

$$\Rightarrow e^{\frac{q}{2a}} = 1 + \frac{P_0 t}{2ma} \Rightarrow \boxed{q = 2a \ln\left(1 + \frac{P_0 t}{2ma}\right)}$$

(3)

$$p = P_0\left(1 + \frac{P_0 t}{2ma}\right) \Rightarrow \boxed{p = P_0 + \frac{P_0^2 t}{2ma}}$$

(4

Problem 3 : $H\left(q, \frac{\partial W}{\partial q}\right) = E$ $\qquad W' := \frac{\partial W}{\partial q}$

$$\boxed{\begin{aligned} W &= W(q, \tilde{q}) \\ \tilde{q} &=: E \\ S &= W - Et \end{aligned}}$$

$$\Rightarrow \frac{1}{2m} e^{-\frac{q}{a}} W'^2 = E$$

$$\Rightarrow W' = \pm \sqrt{2mE}\, e^{+\frac{q}{2a}}$$

$$\Rightarrow W = \pm \sqrt{2mE}\,(-2a)\, e^{\frac{q}{2a}} + W_0$$

$$\Rightarrow W = \mp \sqrt{2mE}\, 2a e^{\frac{q}{2a}} + W_0$$

$$\tilde{p} = -\frac{\partial S}{\partial \tilde{q}} = -\frac{\partial W}{\partial E} + t = \mp \sqrt{\frac{m}{E}}\, 2a e^{\frac{q}{2a}} + W_0 + t$$

$$\Rightarrow e^{\frac{q}{2a}} = \mp \frac{(\tilde{p} - W_0 - t)}{2a} \sqrt{\frac{E}{m}}$$

at $t = 0$, $q = 0$ $\hookrightarrow$ $\mp \frac{\tilde{p} - W_0}{2a}\sqrt{\frac{E}{m}} = 1$

$$\Rightarrow e^{\frac{q}{2a}} = 1 \pm \frac{1}{2a}\sqrt{\frac{E}{m}}\, t$$

$$\Rightarrow \boxed{q = 2a \ln\left[1 \pm \frac{1}{2a}\sqrt{\frac{E}{m}}\, t\right]}$$

$$p = \frac{\partial W}{\partial q} = W' = \pm\sqrt{2mE}\, e^{\frac{q}{2a}}$$

at $t = t_0$, $p = p_0$, $q = 0$ $\hookrightarrow$ $p_0 = \pm\sqrt{2mE}$

$$\Rightarrow \boxed{q = 2a \ln\left[1 + \frac{p_0 t}{2ma}\right]}$$

$$\Rightarrow p = p_0 e^{\frac{q}{2a}} = p_0 \left[1 + \frac{p_0 t}{2ma}\right]$$

$$\Rightarrow \boxed{p = p_0 + \frac{p_0^2 t}{2ma}}$$

Problem 4.a) (i) $p\,dq - H\,dt - (\tilde{p}\,d\tilde{q} - K\,dt) = dF$

Suppose $p, \tilde{q}$ are indep. $q = q(p, \tilde{q}, t)$

$$dq = \frac{\partial q}{\partial p}dp + \frac{\partial q}{\partial \tilde{q}}d\tilde{q} + \frac{\partial q}{\partial t}dt, \quad dF = \frac{\partial F}{\partial \tilde{q}}d\tilde{q} + \frac{\partial F}{\partial p}dp + \frac{\partial F}{\partial t}dt$$

$$\Rightarrow \quad p\frac{\partial q}{\partial p}dp + p\frac{\partial q}{\partial \tilde{q}}d\tilde{q} + p\frac{\partial q}{\partial t}dt - H\,dt - \tilde{p}\,d\tilde{q} + K\,dt$$

$$= \frac{\partial F}{\partial \tilde{q}}d\tilde{q} + \frac{\partial F}{\partial p}dp + \frac{\partial F}{\partial t}dt$$

$$\Rightarrow \begin{cases} p\dfrac{\partial q}{\partial \tilde{q}} - \tilde{p} = \dfrac{\partial F}{\partial \tilde{q}} & (1) \\[2ex] p\dfrac{\partial q}{\partial p} = \dfrac{\partial F}{\partial p} & (2) \\[2ex] p\dfrac{\partial q}{\partial t} + K - H = \dfrac{\partial F}{\partial t} & (3) \end{cases}$$

Let $F_3 := F - pq$

$$\Rightarrow \begin{cases} \dfrac{\partial F_3}{\partial \tilde{q}} = \dfrac{\partial F}{\partial \tilde{q}} - p\dfrac{\partial q}{\partial \tilde{q}} & (4) \\[2ex] \dfrac{\partial F_3}{\partial p} = \dfrac{\partial F}{\partial p} - q - p\dfrac{\partial q}{\partial p} & (5) \\[2ex] \dfrac{\partial F_3}{\partial t} = \dfrac{\partial F}{\partial t} - p\dfrac{\partial q}{\partial t} & (6) \end{cases}$$

(1) & (4) $\Rightarrow$ $\boxed{\tilde{p} = -\dfrac{\partial F_3}{\partial \tilde{q}}}$

$$\boxed{q = -\frac{\partial F_3}{\partial p}}$$

$$K - H = \frac{\partial F_3}{\partial t} \quad \Rightarrow \quad \boxed{K = H + \frac{\partial F_3}{\partial t}}$$

4.b) $$-\frac{\partial F_3}{\partial t} = H\left(-\frac{\partial F_3}{\partial p}, p, t\right)$$

4.c) Let $F_3 = W(\tilde{q}, p) - \tilde{q}t$ for $\tilde{q} = E = \text{const}$

$$\Rightarrow \boxed{E = \tilde{q} = H\left(-\frac{\partial W}{\partial p}, p\right) = \frac{p^2}{2m} + \lambda (W')^4} \qquad W' := \frac{\partial W}{\partial p}$$

$$\Leftrightarrow \quad \boxed{W'^4 = \frac{1}{\lambda}\left(E - \frac{p^2}{2m}\right)}$$