1) Let $\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

1.a) Show that every 2×2 complex matrix can be expressed as a linear combination of these matrices with complex coefficients.

1.b) Show that every 2×2 Hermitian matrix can be expressed as a linear combination of these matrices with real coefficients.

1.c) Compute the spectral representation of σ_1, σ_2, and σ_3 by giving explicit form of the corresponding complete orthonormal set of projection operators.

1.d) Denote the projection operators you find for σ_i by $P_-^{(i)}$, $P_+^{(i)}$, respectively, where the sign corresponds to the eigenvalues ± 1 of σ_i. Compute $P_+^{(i)} P_-^{(j)}$, $P_+^{(i)} P_+^{(j)} P_+^{(i)}$ for all $i, j = 1, 2, 3$.

1.e) Compute $P_+^{(i)} P_+^{(j)} P_+^{(i)}$, and $P_-^{(i)} P_-^{(j)} P_-^{(i)}$ for all $i, j = 1, 2, 3$ with $i \neq j$.

2) Let $\hat{x}, \hat{k} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(\hat{x}\psi)(x) := x\psi(x)$ and $(\hat{p}\psi)(x) := -i\psi'(x)$ where a prime denotes differentiation with respect to x.

2.a) Show that both $\hat{x}$ and $\hat{p}$ are symmetric operators.

2.b) Show that $\hat{p}$ does not have and eigenvalue (or an eigenfunction.)

3) Let $\mathcal{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(\mathcal{P}\psi)(x) := \psi(-x)$ for all $x \in \mathbb{R}$. Show that $\mathcal{P}$ is a self-adjoint operator.

4) Let $\mathscr{H}$ be a separable Hilbert space of dimension larger than one, ψ be a nonzero element of $\mathscr{H}$, and $P := |\psi\rangle\langle\psi|$.

4.a) Show that P is bounded operator.

4.b) Show that P is a symmetric operator.

4.c) Is P self-adjoint? Why?

4.d) Find eigenvalues of P.

5) Let $a \in \mathbb{R}$ and $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(T_a\psi)(x) := \psi(x - a)$ for all $x \in \mathbb{R}$.

5.a) Determine the domain of T_a and show that it is a bounded operator.

5.b) Determine the adjoint of T_a and its domain.

6) Let $a \in \mathbb{R}^+$ and $S_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(S_a\psi)(x) := \psi(ax)$ for all $x \in \mathbb{R}$.

6.a) Determine the domain of S_a and show that it is a bounded operator.

6.b) Determine the adjoint of S_a and its domain.