1) Let $a \in \mathbb{R}$ and $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(T_a\psi)(x) := \psi(x-a)$ for all $x \in \mathbb{R}$. Show that T_a is a unitary operator.

2) Let $a \in \mathbb{R}^+$ and $S_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(S_a\psi)(x) := \psi(ax)$ for all $x \in \mathbb{R}$. Show that S_a is a unitary operator.

3) Let $\mathscr{H}$ be a separable Hilbert space of dimension $N \leq \infty$, $\mathscr{B} = (\psi_n)$ be an orthonormal basis of $\mathscr{H}$, and (u_n) be a sequence of elements of $U(1) := \{e^{i\theta} | \theta \in [0, 2\pi)\}$. Show that the operator $U := \sum_{n=1}^{N} u_n |\psi_n\rangle\langle\psi_n|$ is a unitary operator.

4) Let $\mathscr{H}$ be a separable Hilbert space and $H : \mathscr{H} \to \mathscr{H}$ be a Hermitian operator with a discrete spectrum. Use the spectral representation of H to show that for all $a \in \mathbb{R}$, the operator e^{iaH} is a unitary operator.

5) Let $\vec{n} = (n_1, n_2, n_3)$ be a real unit vector, i.e., $n_i \in \mathbb{R}$ and $\sum_{i=1}^{3} n_i^2 = 1$, and $\sigma_{\vec{n}} = \sum_{i=1}^{3} n_i \sigma_i$, where

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

5.a) Obtain explicit expressions for the eigenvectors $\vec{\psi}_i$ and eigenvalues s_i of $\sigma_{\vec{n}}$.

5.b) Construct the projection matrices $\mathbf{P}_i = \vec{\psi}_i \vec{\psi}_i^{\dagger}$ for all $i \in \{1, 2, 3\}$.

5.c) Check by explicit calculation that the identity $\sigma_{\vec{n}} = \sum_{i=1}^{3} s_i \mathbf{P}_i$ holds.

5.d) Use the spectral representation of $\sigma_{\vec{n}}$ to compute the matrix $e^{\frac{i\theta}{2}\sigma_{\vec{n}}}$ for all $\theta \in \mathbb{R}$ and establish identity: $e^{\frac{ia}{2}\sigma_{\vec{n}}} = \cos\theta\,\sigma_0 + i\sin\theta\,\sigma_{\vec{n}}$.