1) Let $\mathcal{I} := [-1, 1]$, $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$ be the vector space of continuous functions $f : \mathcal{I} \to \mathbb{R}$ having $\mathcal{I}$ as their domain, $o \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$ be the zero function, i.e., $o(x) := 0$ for all $x \in \mathcal{I}$, and $\delta_0 : \mathcal{C}^0(\mathcal{I}, \mathbb{C}) \to \mathbb{C}$ be the everywhere-defined function defined by $\delta_0(f) := f(0)$ for all $f \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$. Suppose that we give $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$ the inner product $\langle\cdot|\cdot\rangle$ defined by $\langle f|g\rangle := \int_{-1}^{1} \overline{f(x)} g(x) dx$, for all $f, g \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$.

- **a.** Show that δ_0 belongs to the algebraic dual of $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$, i.e., it is an everywhere-defined linear operator mapping $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$ to $\mathbb{C}$.
- **b.** For each $n \in \mathbb{Z}^+$, let $f_n \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$ be the function defined by $f_n(x) := e^{-n|x|}$. Show that the sequence $\{f_n\}$ converges to o.
- **c.** Show that the sequence $\{\delta_0(f_n)\}$ does not converge to $\delta_0(o)$.
- **d.** Use parts b and c to show that δ_0 is not a bounded operator.
- **e.** Use parts a and d to show that the topological dual of $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$ is not equal to its algebraic dual.

2) Let $\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

- **a.** Show that every 2×2 complex matrix can be expressed as a linear combination of these matrices with complex coefficients.
- **b.** Show that every 2×2 Hermitian matrix can be expressed as a linear combination of these matrices with real coefficients.
- **c.** Compute the spectral representation of σ_1, σ_2, and σ_3 by giving explicit form of the corresponding complete orthogonal set of projection operators.
- **d.** Denote the projection operators you find for σ_i by $P_-^{(i)}$, $P_+^{(i)}$, respectively, where the sign corresponds to the eigenvalues ± 1 of σ_i. Compute $P_+^{(i)} P_-^{(j)}$, $P_+^{(i)} P_+^{(j)} P_+^{(i)}$ for all $i, j = 1, 2, 3$.
- **e.** Compute $P_+^{(i)} P_+^{(j)} P_+^{(i)}$, and $P_-^{(i)} P_-^{(j)} P_-^{(i)}$ for all $i, j = 1, 2, 3$ with $i \neq j$.

3) Let $(V, \langle\cdot|\cdot\rangle)$ be a complex inner-product space, $P : V \to V$ is a projection operator, and $P^\perp := I - P$, where $I : V \to V$ is the identity operator.

- **a.** Show that $P^\perp$ is also a projection operator.
- **b.** Show that $\phi \in \text{Ran}(P)$ if and only if $P\phi = \phi$.
- **c.** Show that P is an orthogonal projection operator if and only if so is $P^\perp$.
- **d.** Show that if P is an orthogonal projection operator and $\psi \in V$, then $\parallel P\psi \parallel \leq \parallel \psi \parallel$.

e. Show that every orthogonal projection operator acting in V is bounded.

f. Show that every orthogonal projection operator is symmetric.

4) Let $\hat{x}, \hat{k} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(\hat{x}\psi)(x) := x\psi(x)$ and $(\hat{k}\psi)(x) := -i\psi'(x)$ where a prime denotes differentiation with respect to x.

a. Show that both $\hat{x}$ and $\hat{k}$ are symmetric operators.

b. Show that $\hat{k}$ does not have an eigenvalue (or an eigenvector.)

5) Let $\mathcal{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(\mathcal{P}\psi)(x) := \psi(-x)$ for all $x \in \mathbb{R}$. Show that $\mathcal{P}$ is a self-adjoint operator.

6) Let $\mathscr{H}$ be a separable Hilbert space of dimension larger than one, ψ be a nonzero element of $\mathscr{H}$, and $P_\psi := |\psi\rangle\langle\psi|$.

a. Show that P_ψ is bounded operator.

b. Show that P_ψ is a symmetric operator.

c. Is P_ψ self-adjoint? Why?

d. Find eigenvalues of P_ψ.

7) Let $a \in \mathbb{R}$ and $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(T_a\psi)(x) := \psi(x - a)$ for all $x \in \mathbb{R}$.

a. Determine the domain of T_a and show that it is a bounded operator.

b. Determine the adjoint of T_a and its domain.

8) Let $a \in \mathbb{R}^+$ and $S_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $(S_a\psi)(x) := \psi(ax)$ for all $x \in \mathbb{R}$.

a. Determine the domain of S_a and show that it is a bounded operator.

b. Determine the adjoint of S_a and its domain.