I) Consider a particle of mass m that moves on the real line and interacts with the potential:

$$v(x) := \left\{ \begin{array}{cc} \lambda \sin^2(x/a) & \text{for} \quad 0 < x < \pi a/4, \\ 0 & \text{otherwise,} \end{array} \right.$$

where a and λ are real and positive parameters.

I.1) Use WKB approximation to obtain a position wave function $\psi(x)$ satisfying the time-independent Schrödinger equation with energy $E = \lambda$. You must give the explicit form of $\psi(x)$ for all $x \in \mathbb{R}$ up to a normalization constant.

I.2) Find an explicit condition on λ and a under which your solution to part I.1 is a reliable approximate solution of the time-independent Schrödinger equation.

II) Consider a particle of mass m that moves on the real line and interacts with the potential:

$$v(x) := \left\{ \begin{array}{ccc} -\lambda & \text{for} & -a < x < 0, \\ \lambda & \text{for} & 0 \leq x < a, \\ 0 & \text{for} & |x| \geq a, \end{array} \right.$$

where a and λ are real and positive parameters. Suppose that at time $t = 0$ the particle is in the state defined by the position wave function: $\psi(x) := e^{-(\alpha+i\beta)|x-b|}$, where α, β and b are real numbers and $\alpha > 0$.

II.1) Calculate the probability of finding this particle in the region defined by $|x| \leq a$.

II.2) Calculate the current density of this particle for $x \neq b$.

II.3) Obtain an explicit expression for the entries of the transfer matrix of this potential.

II.4) Find an explicit expression for a function $F : \mathbb{C} \to \mathbb{C}$ whose real and negative zeros give the energy of the bound states of the system.

II.5) Find a condition on λ and a under which the system has a single bound state and determine its energy.