Phys 517: Midterm Exam
Fall 2016

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 3 hours.
- You are allowed to consult Srednicki's QFT.

Problem 1 (15 points) Show that a 4-vector $a = (a^0, \vec{a})$ is space-like if and only if there is a Lorentz boost that maps it to a 4-vector of the form $(0, \vec{b})$. Find $\vec{b}$ as a function of a^0 and $\vec{a}$. Recall that under the Lorentz boost determined by the velocity $\vec{v}$, every 4-vector $x = (x^0, \vec{x})$ transforms according to $x^0 \to \dfrac{x^0 + \vec{v}\cdot\vec{x}}{\sqrt{1-\vec{v}^2}}$ and $\vec{x} \to \dfrac{\vec{x} + x^0\vec{v}}{\sqrt{1-\vec{v}^2}}$.

a is space-like $\iff -(a^0)^2 + \vec{a}^2 > 0$ ①

If $a^0 = 0$, a is space-like & identity transformation maps a to $(0, \vec{b})$ with $\vec{b} = \vec{a}$.

If $a^0 \neq 0$, ① $\iff \dfrac{|\vec{a}|}{|a^0|} > 1$

($\Rightarrow$) Suppose that a is spacelike. Then $\vec{a} \neq \vec{0}$ & we can choose $\vec{v} = -\dfrac{a^0}{|\vec{a}|^2}\vec{a}$. Clearly $|\vec{v}| = \dfrac{|a^0|}{|\vec{a}|} < 1$ and

$$\Rightarrow \begin{cases} a^0 \to \dfrac{a^0 + \vec{v}\cdot\vec{a}}{\sqrt{1-\vec{v}^2}} = \dfrac{a^0 - a^0}{\sqrt{1-v^2}} = 0 \checkmark \\ \vec{a} \to \vec{b} := \dfrac{\vec{a} + a^0\vec{v}}{\sqrt{1-\vec{v}^2}} = \dfrac{\vec{a} - \dfrac{(a^0)^2\vec{a}}{|\vec{a}|^2}}{\sqrt{1 - \dfrac{(a^0)^2}{|\vec{a}|^2}}} \end{cases}$$

$$= \frac{[|\vec{a}|^2 - (a^0)^2]\vec{a}}{|\vec{a}|\sqrt{|\vec{a}|^2 - (a^0)^2}} = \sqrt{-(a^0)^2 + |\vec{a}|^2}\,\frac{\vec{a}}{|\vec{a}|}$$

($\Leftarrow$) Suppose that $\exists$ a Lorentz boost that maps a to $(0, \vec{b})$. Then $-(a^0)^2 + \vec{a}^2 = a\cdot a = (0, \vec{b})\cdot(0, \vec{b}) = |\vec{b}|^2 > 1 \Rightarrow$ a is spacelike. ▨

Problem 2 (15 points) Let $\mathbf{J}_i$ and $\mathbf{K}_i$ be the generators of the Lorentz group in its standard representation, i.e., for all $\mu \in \{0,1,2,3\}$ and $i,j,k \in \{1,2,3\}$,

$$(\mathbf{J}_i)_{00} = (\mathbf{J}_i)_{\mu 0} = (\mathbf{J}_i)_{0\mu} = 0, \qquad (\mathbf{J}_i)_{jk} = -i\hbar\epsilon_{ijk},$$
$$(\mathbf{K}_i)_{00} = (\mathbf{K}_i)_{jk} = 0, \qquad (\mathbf{K}_i)_{j0} = (\mathbf{K}_i)_{0j} = -i\hbar\delta_{ij}.$$

Use these relations to express $[\mathbf{K}_i, \mathbf{K}_j]$ in terms of $\mathbf{J}_k$ and/or $\mathbf{K}_k$.

$$[\mathbf{K}_i, \mathbf{K}_j]_{00} = (\mathbf{K}_i)_{0\mu}(\mathbf{K}_j)_{\mu 0} - i \leftrightarrow j$$
$$= (\mathbf{K}_i)_{0k}(\mathbf{K}_j)_{k0} - i \leftrightarrow j$$
$$= (-i\hbar)^2 \delta_{ik}\delta_{jk} - i \leftrightarrow j$$
$$= (-i\hbar)^2 \delta_{ij} - i \leftrightarrow j \quad = 0 \qquad (1)$$

$$[\mathbf{K}_i, \mathbf{K}_j]_{0k} = (\mathbf{K}_i)_{0\nu}(\mathbf{K}_j)_{\nu k} - i \leftrightarrow j$$
$$= (\mathbf{K}_i)_{0\ell}\underbrace{(\mathbf{K}_j)_{\ell k}}_{0} - i \leftrightarrow j \quad = 0 \qquad (2)$$

$$[\mathbf{K}_i, \mathbf{K}_j]_{k0} = (\mathbf{K}_i)_{k\nu}(\mathbf{K}_j)_{\nu 0} - i \leftrightarrow j$$
$$= (\mathbf{K}_i)_{k0}\underbrace{(\mathbf{K}_j)_{00}}_{0} - i \leftrightarrow j \quad = 0 \qquad (3)$$

$$[\mathbf{K}_i, \mathbf{K}_j]_{k\ell} = (\mathbf{K}_i)_{k\mu}(\mathbf{K}_j)_{\mu\ell} - i \leftrightarrow j$$
$$= (\mathbf{K}_i)_{k0}(\mathbf{K}_j)_{0\ell} - i \leftrightarrow j$$
$$= (-i\hbar)^2 \delta_{ik}\delta_{j\ell} - i \leftrightarrow j$$
$$= (-i\hbar)^2 [\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}]$$
$$= (-i\hbar)^2 \epsilon_{mij}\epsilon_{mk\ell}$$
$$= -i\hbar\,\epsilon_{ijm}\underbrace{(-i\hbar\epsilon_{mk\ell})}_{(\mathbf{J}_m)_{k\ell}}$$
$$= -i\hbar\,\epsilon_{ijm}(\mathbf{J}_m)_{k\ell} \qquad (4)$$

$$(1) - (4) \Rightarrow \quad [\mathbf{K}_i, \mathbf{K}_j]_{\mu\nu} = (-i\hbar\,\epsilon_{ijm}\mathbf{J}_m)_{\mu\nu}$$
$$\Downarrow$$
$$[\mathbf{K}_i, \mathbf{K}_j] = -i\hbar\,\epsilon_{ijm}\mathbf{J}_m .$$

Problem 3 (15 points) Let ϕ be a free real scalar field, $a(\vec{k})$ is the associated annihilation operator, so that

$$\phi(x) = \int_{\mathbb{R}^3} \widetilde{d^3\vec{k}} \left[e^{ik\cdot x} a(\vec{k}) + e^{-ik\cdot x} a(\vec{k})^\dagger \right],$$

and $|\vec{k}\rangle$ be a one-particle state vector with momentum $\vec{k}$. Derive the transformation rule for $a(\vec{k})$ and $|\vec{k}\rangle$ under the action of a proper orthochronous Poincaré transformation (Λ, a), i.e., $x \xrightarrow{(\Lambda,a)} \Lambda x + a$.

$\phi \to \tilde{\phi}$ when $\tilde{\phi}(x) := \phi(\Lambda x + a)$ ①

$$\tilde{\phi}(x) = \int_{\mathbb{R}^3} \widetilde{d^3\vec{k}} \left[e^{ikx} \tilde{a}(\vec{k}) + e^{-ikx} \tilde{a}(\vec{k})^\dagger \right] \quad ②$$

Also ① $\Rightarrow$

$$\tilde{\phi}(x) = \int_{\mathbb{R}^3} \widetilde{d^3\vec{k}} \left[e^{ik\cdot(\Lambda x + a)} a(\vec{k}) + e^{-ik\cdot(\Lambda x + a)} a(\vec{k})^\dagger \right] \quad ③$$

$$k\cdot(\Lambda x + a) = \underbrace{k\cdot(\Lambda x)}_{(k,\Lambda x) = (\underbrace{\Lambda^{-1}k}_{\tilde{k}}, x)} + k\cdot a = \Lambda^{-1}k\cdot x + k\cdot a = \tilde{k}\cdot x + k\cdot a$$

Also $k\cdot a = (k, a) = (\Lambda^{-1}k, \Lambda^{-1}a) = (\tilde{k}, \Lambda^{-1}a) = \tilde{k}\cdot\Lambda^{-1}a$

$$\Rightarrow k\cdot(\Lambda x + a) = \tilde{k}\cdot x + \tilde{k}\cdot\Lambda^{-1}a = \tilde{k}\cdot(x + \Lambda^{-1}a) \quad ④$$

③ & ④ $\Rightarrow$

$$\tilde{\phi}(x) = \int_{\mathbb{R}^3} \widetilde{d^3\vec{k}} \left[e^{i\tilde{k}(x+\Lambda^{-1}a)} a(\vec{k}) + e^{-i\tilde{k}(x+\Lambda^{-1}a)} a(\vec{k})^\dagger \right]$$

" measure is inv. under (Λ, a)

$$= \int_{\mathbb{R}^3} \widetilde{d^3\vec{\tilde{k}}} \left[e^{i\tilde{k}\cdot x} \left(e^{i\tilde{k}\cdot\Lambda^{-1}a} a(\Lambda\vec{\tilde{k}}) \right) + e^{-i\tilde{k}x} \left(e^{-i\tilde{k}\cdot\Lambda^{-1}a} a(\Lambda\vec{\tilde{k}})^\dagger \right) \right]$$

Relabel: $\vec{\tilde{k}} \to \vec{k}$

$$= \int_{\mathbb{R}^3} \widetilde{d^3\vec{k}} \left[e^{ik\cdot x} \left(e^{ik\cdot\Lambda^{-1}a} a(\Lambda\vec{k}) \right) + e^{-ik\cdot x} \left(e^{-ik\,\Lambda^{-1}a} a(\Lambda\vec{k})^\dagger \right) \right] \quad ⑤$$

② & ⑤ $\Rightarrow$ $\boxed{a(\vec{k}) \to \tilde{a}(\vec{k}) = e^{ik\cdot\Lambda^{-1}a} a(\Lambda\vec{k})}$

$$\boxed{|\vec{k}\rangle \to \tilde{a}(\vec{k})^\dagger|0\rangle = e^{-ik\cdot\Lambda^{-1}a} a(\Lambda\vec{k})^\dagger|0\rangle = e^{-ik\cdot\Lambda^{-1}a} |\Lambda\vec{k}\rangle}$$

Problem 4 Consider computing $\langle 0|\mathscr{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle$ in the ϕ^3-theory.

4.a (10 points) Find the Feynman diagram(s) that give the leading order term in the perturbative expansion of $\langle 0|\mathscr{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle$ in powers of the coupling constant g. Justify your response.

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle = \left[\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\left(-i\frac{\delta}{\delta J(x_3)}\right) e^{iW_1''[J]}\right]\Big|_{J=0}$$

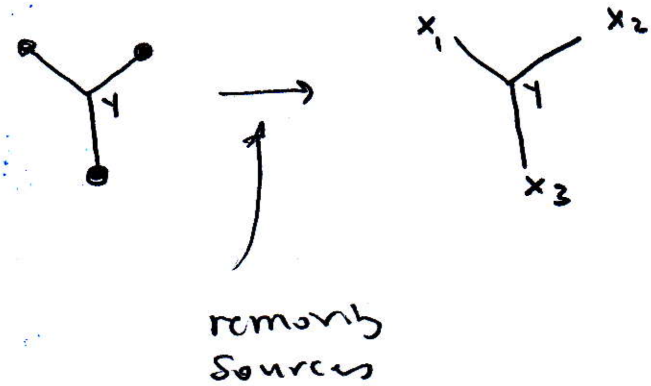
$$= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\left(-i\frac{\delta}{\delta J(x_3)}\right)\left(1 + iW_1''[J] + \frac{(iW_1''[J])^2}{2!} + \dots\right)\Big|_{J=0}$$

Because $iW_1''[J] = \sum_I C_I$ → Connected graphs with at least 2 sources and no tadpoles

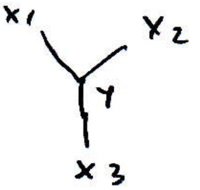
= Sum of the contribution of connected graphs having 3 sources and no tadpoles with sources removed

$E = 3 = 2P - 3V - 2V'$

lowest order in $g \leftrightarrow V = 1$ & $V' = 0 \Rightarrow P = 3$



4.b) (15 points) Use the formula $\Delta(x) := \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2+m^2}$ for the Feynman propagator to compute the contribution of the diagram(s) you find in part **a** of this problem.



$$= i Z_g g \int d^4y \; \frac{\Delta(x_1-y)}{i} \frac{\Delta(x_2-y)}{i} \frac{\Delta(x_3-y)}{i}$$

$$= -Z_g g \int d^4y \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \frac{e^{i[k_1(x_1-y)+k_2(x_2-y)+k_3(x_3-y)]}}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)}$$

$$\int d^4y \; e^{-iy(k_1+k_2+k_3)} = (2\pi)^4 \delta^4(k_1+k_2+k_3)$$

$$= -Z_g g \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{i[k_1x_1+k_2x_2-(k_1+k_2)x_3]}}{(k_1^2+m^2)(k_2^2+m^2)[(k_1+k_2)^2+m^2]}$$

$$= -Z_g g \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{i[k_1(x_1-x_3)+k_2(x_2-x_3)]}}{(k_1^2+m^2)(k_2^2+m^2)[(k_1+k_2)^2+m^2]}$$

$$= \int d^4k_1 \int d^4k_2 \; \underbrace{\left(\frac{-Z_g g}{(2\pi)^8}\right) \frac{e^{i[k_1(x_1-x_3)+k_2(x_2-x_3)]}}{(k_1^2+m^2)(k_2^2+m^2)[(k_1+k_2)^2+m^2]}}_{F(k_1,k_2,x_1,x_2,x_3)}$$

4.c (10 points) Give the Feynman diagrams that contribute to the next to leading order term in the perturbative expansion of $\langle 0|\mathscr{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle$. If possible, include the diagrams involving the two-point vertices.

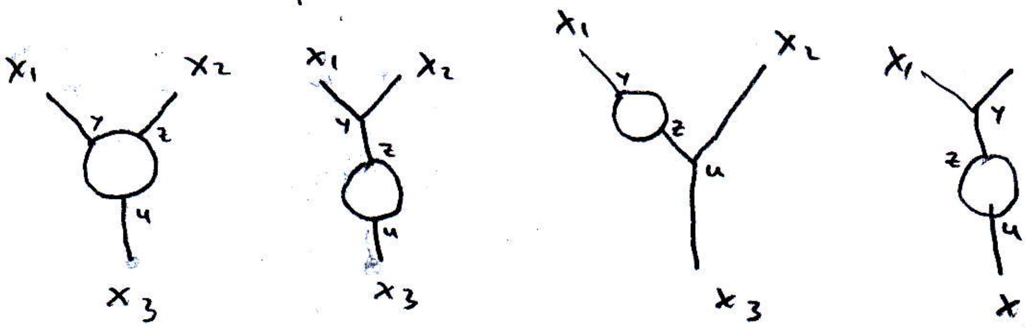
$E = 3 = 2P - 3V - 2V'$ $\qquad$ $V = \mathcal{O}(s)$, $\quad V' = \mathcal{O}(s^2)$

$\Rightarrow$ Take $V = 3, V' = 0$, $\quad$ or $\quad V = 1, V' = 1$

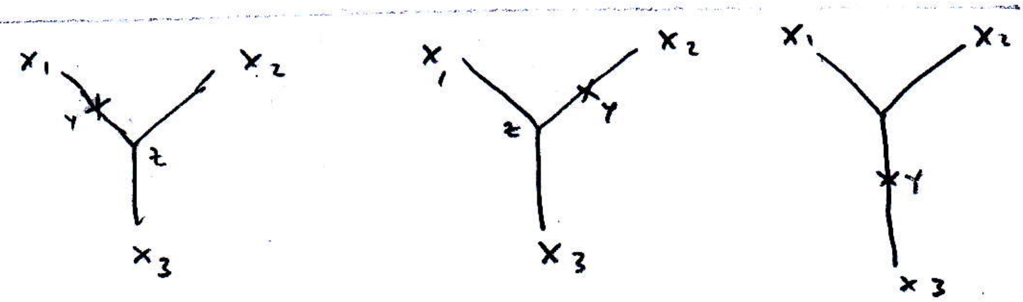
$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$P = 6$ $\qquad\qquad\qquad\qquad$ $P = 4$

$V = 3$
$V' = 0$

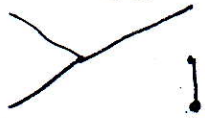


$V = 1, V' = 1$

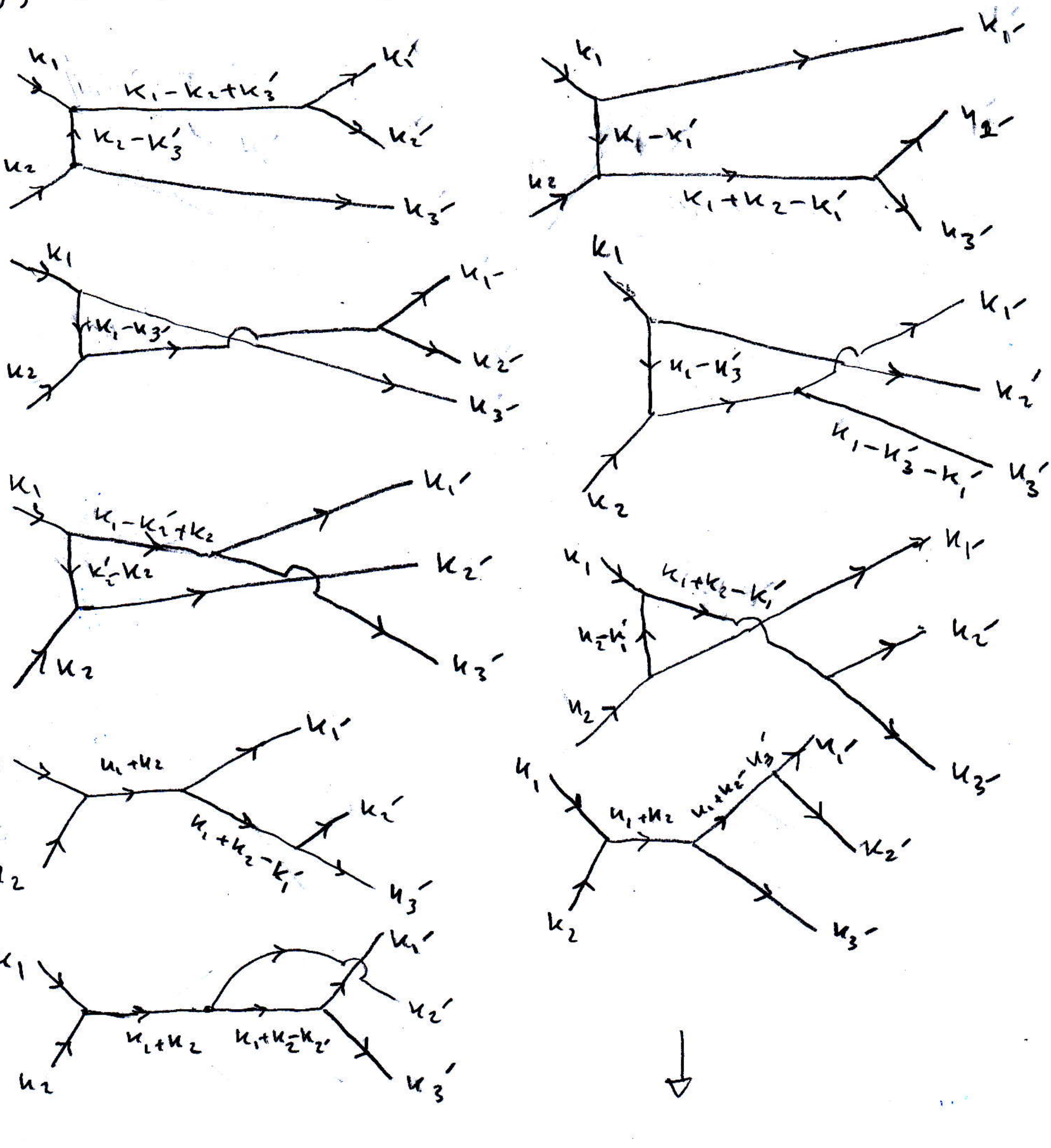


Problem 5 (20 points) Consider a process in which two incoming particles (with momenta k_1 and k_2) scatter and produce three outgoing particles (with momenta $k_{1'}$, $k_{2'}$, and $k_{3'}$). Suppose that we wish to calculate the leading order term in the perturbative calculation of the scattering (transfer) matrix element $\mathcal{T}$ for this process in the ϕ^3-theory. Draw all the (momentum space) Feynman diagrams that enter this calculation. Label the links of these diagrams properly and specify their orientation.

$$E = 5 = 2P - 3V - 2V' \implies V \text{ is odd}$$

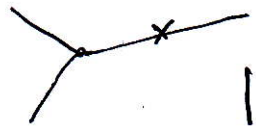
$O(g)$ $V = 1$, $V' = 0$ $\implies$ $P = 4$: ] This is disconnected

$O(g^3)$ $V = 3$, $V' = 0$ $\implies$ $P = 7$:



Note that for $\mathcal{O}(s^3)$ we can also set

<u>$V=1,\ V'=1$:</u> $\quad 5 = 2P - 3V - 2V' \Rightarrow P = 5$

 $\Rightarrow$ There are also disconnected.