1 (5 points) Use the identity $\boldsymbol{\Lambda}^T\boldsymbol{\eta}\boldsymbol{\Lambda} = \boldsymbol{\eta}$ to prove that the Lorentz matrices $\boldsymbol{\Lambda}$ form a subgroup of $GL(4,\mathbb{R})$, i.e., show that the inverse of any Lorentz matrix and the product of any two Lorentz matrices are Lorentz matrices.

2 Let $\mathcal{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the parity operator defined by $\mathcal{P}\psi(x) = \psi(-x)$ and

$$\langle\!\langle \phi, \psi \rangle\!\rangle = \langle \phi | e^{-\kappa \mathcal{P}} \psi \rangle,$$

where $\langle \cdot | \cdot \rangle$ is the standard L^2-inner product, and $\kappa \in \mathbb{R}$.

2a (3 points) Show that $e^{-\kappa\mathcal{P}} = \cosh(\kappa) I - \sinh(\kappa)\mathcal{P}$.

2b (3 points) Show that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is a positive-definite inner product on $L^2(\mathbb{R})$.

2c (4 points) Let $\tilde{\mathscr{H}}$ be the Hilbert space obtained by endowing $L^2(\mathbb{R})$ with $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Show that the standard position and momentum operator that are defined by $X\psi(x) := x\psi(x)$ and $P\psi(x) := -\hbar\partial_x\psi(x)$ do not act as Hermitian operators in $\tilde{\mathscr{H}}$, but that P^2 does.

2d (6 points) Let $\tilde{X} := X\, e^{-\kappa\mathcal{P}}$ and $\tilde{P} := P\, e^{-\kappa\mathcal{P}}$. Show that the following relations hold.

$$\langle\!\langle \phi, \tilde{X}\psi \rangle\!\rangle = \langle\!\langle \tilde{X}\phi, \psi \rangle\!\rangle, \qquad \langle\!\langle \phi, \tilde{P}\psi \rangle\!\rangle = \langle\!\langle \tilde{P}\phi, \psi \rangle\!\rangle, \qquad [\tilde{X}, \tilde{P}] = i\hbar 1,$$

where 1 is the identity operator acting in $\tilde{\mathscr{H}}$.

2e (5 points) Use $\tilde{X}$ and $\tilde{P}$ to represent the position and momentum operators for a free particle determine by the Hilbert space $\tilde{\mathscr{H}}$ and the Hamiltonian $H = P^2/(2m)$. Show that the localized state vector $\tilde{\delta}_y$ of this particle, that is localized at y, is given by

$$\tilde{\delta}_y(x) = \cosh(\tfrac{\kappa}{2})\delta(x-y) + \sinh(\tfrac{\kappa}{2})\delta(x+y).$$

2f (4 points) Show that the following identities hold.

$$\langle\!\langle \tilde{\delta}_x, \tilde{\delta}_y \rangle\!\rangle = \delta(x-y), \qquad \int_{-\infty}^{\infty} dx\, \langle\!\langle \tilde{\delta}_x, \psi \rangle\!\rangle\, \tilde{\delta}_x = \psi.$$

2g (5 points) Give the expression for the position wave function $\tilde{f}(x,t)$ of an evolving state vector $\psi(t) \in \tilde{\mathscr{H}}$, i.e., $\langle\!\langle \tilde{\delta}_x, \psi(t) \rangle\!\rangle$, and the probability density $\tilde{\rho}(x,t)$ for the localization of the particle in this state in terms of $\psi(x,t) := \langle \delta_x | \psi(t) \rangle$.

Reference: J. Phys. A **39** 10171 (2006), Section 5.

3 (5 points) Show that the D'Alembertian operator, i.e., $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$, is invariant under Poincaré transformations.

4-6 (30 points) Solve Problems 1.1, 1.2, and 1.3 on page 14 of Srednicki.

7 (30 points) Read pages 58-62 of Weinberg's "The Quantum Theory of Fields," Volume 1 (Cambridge University Press, 1995), and give the details of the derivation of the Poincaré algebra relations, i.e., Eqs. 2.4.18 - 2.4.24 on page 61 of this reference.