we have, finally,

$$\begin{aligned} I &= (2\pi i)^{(m+n)/2}(\det A_B)^{-\frac{1}{2}}(\det B_B)^{\frac{1}{2}}J^{-1} \\ &= (2\pi i)^{(m+n)/2}(\det A)^{-\frac{1}{2}}[\det(B + C^{\sim}A^{-1}C)]^{\frac{1}{2}} \\ &= (2\pi i)^{(m+n)/2}(\operatorname{sdet} M)^{-\frac{1}{2}}. \end{aligned} \tag{1.7.64}$$

The rule for determining the phase of the body of $(\operatorname{sdet} M)^{-\frac{1}{2}}$ is the following: A factor $-i$ for each λ having negative body. A factor i for each μ having positive body and a factor $-i$ for each μ having negative body, the μ's being determined by (1.7.57) with O_2 restricted to be an orthogonal matrix of *positive* determinant.

It should be noted that the exact analogy with the classical Gaussian integral, displayed by eq. (1.7.64), depends on the choice (1.3.15) for the *c*-number Z and the choice (1.3.19) for the *a*-number contribution to the volume element, choices that have already been seen to maintain an exact analogy in Fourier transform theory. Note also that eq. (1.7.64) holds even when n is odd, for in that case both I and $\det(B + C^{\sim}A^{-1}C)$ vanish.

Exercises

1.1 Prove that (1.1.11) is the general solution of eq. (1.1.10). *Hint*: Expand v in the form (1.1.9) and $f(v)$ in the form (1.1.2), (1.1.3). Regard the coefficients of the latter expansion as functions of the c's in eq. (1.1.9). Vary the c's infinitesimally, obtaining the general form for dv. Find the conditions on the coefficients in the expansion of $f(v)$ that are necessary for df to factorize as in eq. (1.1.10). Show that these conditions lead to (1.1.11).

1.2 Outline a proof that if f is superanalytic on a smooth simply connected 2-dimensional surface-with-boundary in $\mathbf{C}_c$ then $\oint f(u)du = 0$ where the integral is over the boundary. *Hint*: Approximate the surface by a simplicial decomposition into small triangles. Show that the integral over the boundary of each triangle vanishes to second order in small quantities. Pass to the limit in which the dimensions of the triangles tend to zero. (Leave rigor to the specialists.)

1.3 Starting with the result of the preceding exercise develop the elementary parts of the theory of superanalytic functions on $\mathbf{C}_c$ in complete parallel with the theory of ordinary analytic functions of a complex variable. In particular, develop the theory of Taylor and Laurent series and show that (1.1.18), where $f_{a_1 \ldots a_n}(u)$ are functions of the form (1.1.17), is the general solution of eq. (1.1.15).

1.4 Show that $\oint f(v)\,\mathrm{d}v$, where $f(v)$ has the general form (1.1.11), is not independent of the contour and does not generally vanish.

1.5 Show that the fundamental theorem of algebra does not hold in $\mathbf{C}_c$. Give examples of finite-order polynomials in u, with coefficients in $\mathbf{C}_c$, that do not have any zeros. Give examples of finite-order polynomials that have infinitely many different zeros even when their coefficients are in $\mathbf{C}$, i.e., are ordinary complex numbers. (Contributed by F.L. Newman.)

1.6 If a complex c-number u has nonvanishing body, one may define its *absolute value* by $|u| \overset{\text{def}}{=} (u^*u)^{\frac{1}{2}}$, the square root with positive body being understood. Prove that this square root is unique. Prove that every complex c-number with nonvanishing body can be expressed in the form $u = |u|e^{i\phi}$ where ϕ is a real c-number.

1.7 Derive eq. (1.7.2) where f is an arbitrary differentiable function of the x^i, with values in Λ_∞.

1.8 Let V be a (k,l)-dimensional supervector space and let $\{L_i\}$ be a set of $(k,l)\times(k,l)$ matrices (i.e., having the block structure of the matrix K of eq. (1.4.40)) which act on V. Denote by Ker $\{L_i\}$ the *set* of sub-supervector spaces of V that remain invariant under the actions of $\{L_i\}$. The set $\{L_i\}$ is said to be *irreducible* if every element of Ker$\{L_i\}$ has the form αV for some pure supernumber α.

Let $\{L_i\}$ be an irreducible set of $(k,l)\times(k,l)$ matrices and let $\{M_i\}$ be an irreducible set of $(m,n)\times(m,n)$ matrices that can be put into one-to-one correspondence with the set $\{L_i\}$. Suppose there exists a $(k,l)\times(m,n)$ matrix A such that $L_iA = AM_i$ for all i. Prove the analog of Schur's lemma for supervector spaces. That is, prove that either $A=0$ or else $k=m$, $l=n$ and A is a nonsingular matrix times a pure supernumber. Prove, as a corollary, that all nonsingular $(k,l)\times(k,l)$ matrices B satisfying $L_iB = BL_i$ for all i, necessarily have the form $B = \lambda 1_{(k,l)}$ where λ is a c-number with nonvanishing body.

1.9 The $(m,n)\times(m,n)$ matrix K of eq. (1.4.40) maps c-type supervectors into c-type supervectors and a-type supervectors into a-type supervectors. It may therefore be called a c-type matrix. One may also consider a-type matrices, which map c-type supervectors into a-type supervectors and vice versa. If, on the right-hand side of eq. (1.4.40), the submatrices A and B had a-type elements and the submatrices C and D had c-type elements then K would be an a-type matrix.

One may generalize the supertrace and supertranspose so that they apply to a-type matrices. This is done by defining

$$\operatorname{str} K \stackrel{\text{def}}{=} (-1)^{i(K+1)}{}_i K^i, \qquad \operatorname{str} L \stackrel{\text{def}}{=} (-1)^{i(L+1)}{}^i L_i,$$

$$ {}^i K^{\sim}{}_j \stackrel{\text{def}}{=} (-1)^{j(i+1)+K(i+j)}{}_j K^i, \qquad {}_i L^{\sim}{}^j \stackrel{\text{def}}{=} (-1)^{i(j+1)+L(i+j)}{}^j L_i,$$

$$ {}_i M^{\sim}{}_j \stackrel{\text{def}}{=} (-1)^{i+j+ij+M(i+j)}{}_j M_i, \qquad {}^i N^{\sim}{}^j \stackrel{\text{def}}{=} (-1)^{ij+N(i+j)}{}^j N^i.$$

Show that these definitions yield

$$K^{\sim\sim} = K, \quad L^{\sim\sim} = L, \quad M^{\sim\sim} = M, \quad N^{\sim\sim} = N, \quad \operatorname{str} K^{\sim} = \operatorname{str} K, \quad \operatorname{str} L^{\sim} = \operatorname{str} L,$$

and, for all admissible combinations of index positions,

$$(PQ)^{\sim} = (-1)^{PQ} Q^{\sim} P^{\sim},$$
$$\operatorname{str}(PQ) = (-1)^{PQ} \operatorname{str}(QP).$$

1.10 If $m = n$ one may generalize the superdeterminant so that it too applies to a-type matrices. One must then, however, distinguish between *left* and *right* superdeterminants, defined respectively by

$$\delta \ln \operatorname{sdet}_{\mathrm{L}} M = \operatorname{str}(\delta M M^{-1}),$$
$$\delta \ln \operatorname{sdet}_{\mathrm{R}} M = \operatorname{str}(M^{-1} \delta M) = (-1)^M \delta \ln \operatorname{sdet}_{\mathrm{L}} M.$$

Show that these superdeterminants satisfy

$$\operatorname{sdet}_{\mathrm{R}} M = (\operatorname{sdet}_{\mathrm{L}} M)^{(-1)^M},$$
$$\operatorname{sdet}_{\mathrm{L}}(LM) = (\operatorname{sdet}_{\mathrm{L}} L)(\operatorname{sdet}_{\mathrm{L}} M)^{(-1)^L},$$
$$\operatorname{sdet}_{\mathrm{R}}(LM) = (\operatorname{sdet}_{\mathrm{R}} L)^{(-1)^M}(\operatorname{sdet}_{\mathrm{R}} M).$$

If the matrix (1.4.40) is a-type show that its left and right superdeterminants are given by

$$\operatorname{sdet}_{\mathrm{L}}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \left[\operatorname{sdet}_{\mathrm{R}}\begin{pmatrix} A & C \\ D & B \end{pmatrix}\right]^{-1}$$
$$= [\det(C - AD^{-1}B)](\det D)^{-1} = (\det C)[\det(D - BC^{-1}A)]^{-1}$$
$$= (\det C)(\det D)^{-1}[\det(1_m - C^{-1}AD^{-1}B)]$$
$$= (\det C)(\det D)^{-1}[\det(1_m - D^{-1}BC^{-1}A)]^{-1}.$$

These last equations can, in fact, be regarded as defining the left and right superdeterminants in the yet more general case in which A, B, C, D have the dimensions $m \times n$, $n \times m$, $m \times m$, $n \times n$ respectively, with $m \neq n$, A and B having a-type elements and C and D having c-type elements.