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Title: Uncertainty and Randomness in Pseudo-Hermitian Quantum Systems

Abstract: Recent developments in quantum information have catalyzed a vigorous debate on the feasibility and implications of Non-Hermitian quantum mechanics (NHQM), particularly concerning non-Hermitian or pseudo-Hermitian Hamiltonians. Divergent from the orthodox conception of Hermitian Hamiltonians, which provide a foundational pillar in conventional quantum mechanics due to their guaranteed natural energy eigenvalues, the exploration into non-Hermitian Hamiltonians represents a daring foray into a domain replete with unforeseen properties and promising applications.

Non-Hermitian Hamiltonians, in the broadest sense, are not restricted to real eigenvalues, thereby permitting a wide range of complex energy spectra and challenging the fundamental principles of quantum information theory. Significant concerns surrounding the nonphysical implications of NHQM, such as the violation of the no-signalling theorem, discrimination of nonorthogonal states, and the abnormal increase of quantum entanglement through local operations, were expressed [1]. These critiques question the viability of NHQM as a foundational theory and its aptitude in quantum information processing. However, recent research has introduced a reformed formulation of NHQM based on the geometry of Hilbert spaces [1]. This novel approach has been shown to reconcile the apparent contradictions, demonstrating that core quantum principles, such as the no-cloning and no-deleting theorems, can indeed be preserved in finite-dimensional non-Hermitian quantum systems, including parity-time symmetric and pseudo-Hermitian cases. Consequently, this remodelled understanding of NHQM brings us closer to reconciling these challenges and opens the door for potentially revolutionary applications.

One essential application lies in quantum information, a field that seeks to exploit quantum phenomena for information processing and communication. In quantum computation, non-Hermitian Hamiltonians can enable more efficient algorithms, leveraging their unique properties, such as nonorthogonality and complex energy spectra. Furthermore, quantum communication could be enhanced by exploiting non-Hermitian Hamiltonians for quantum key distribution, offering increased resistance to eavesdropping and improved noise resilience.

A salient and intriguing application of non-Hermitian Hamiltonians pertains to randomness generation. True randomness is vital in various scientific disciplines, including cryptography, computer simulations, and statistical sampling. The complexity and nonorthogonality exhibited by non-Hermitian systems can offer an unprecedented level of randomness, potentially overcoming the limitations of conventional pseudorandom number generators. Moreover, the study revealed intricate quantum correlations, specifically squeezing, near the critical point of the phase transition. Such quantum correlations, inherent to non-Hermitian systems, increase the complexity of the system state and, therefore, can contribute to enhanced randomness. Exploiting these phenomena for randomness generation could mitigate the inherent predictability of conventional pseudorandom number generators, offering a more secure and robust randomness source [5,6]. This could have profound implications for fields heavily relying on random numbers, including cryptography and Monte Carlo simulations.

One can model the quantum system using a mathematical construct known as a pseudo-

Hermitian matrix [2,3,4] as follows:

$$A^\dagger = \eta A \eta^{-1} \tag{1}$$

where $A^\dagger$ is the Hermitian adjoint of A, and η is a Hermitian operator. Unlike regular matrices, these special matrices can have complex numbers as their eigenvalues, a kind of 'characteristic value' associated with the matrix. Despite this, they retain some essential properties of regular matrices, making them useful for describing certain quantum systems. The quantum system's behaviour is controlled by two parameters, referred to as b and c, in the study. These parameters influence whether the eigenvalues of the matrix are real or complex numbers. The distinction between natural and complex eigenvalues significantly affects the quantum system's behaviour. The matrix $A2$ is defined as:

$$A2 = \hat{U} + b\hat{T}\sqrt{\hat{U}} - ic\hat{S} \tag{2}$$

The eigenvalues of $A2$ are real if $b \geq c$ and complex conjugate if $b < c$. This indicates that $A2$ is pseudo-Hermitian concerning the metric P-Q. The researchers then calculate a quantity known as the von Neumann entropy of the system. This is a measure of uncertainty or 'ignorance' about the quantum system's state. In other words, it quantifies how much we don't know about the system. The equation gives the entropy:

$$S = -Tr(\rho \log \rho) \tag{3}$$

where ρ is the density matrix of the system, the entropy is found to depend on the parameters b and c, as well as time. Interestingly, the study finds that the system's entropy can oscillate over time, or stabilize at a fixed value, depending on the values of b and c. This suggests that the system can generate a source of randomness, as the entropy represents the uncertainty about the system's state. The changing entropy over time could be used to generate random numbers.

In conclusion, by studying the uncertainty and entanglement properties of a system modelled by pseudo-Hermitian matrices, one can show that the inherent uncertainty of quantum mechanics can be harnessed as a source of randomness. This could have potential applications in cryptography, where true randomness is highly valuable.

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