# Robust Turbo Equalization Under Channel Uncertainties

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***Abstract*— Robust turbo equalization over discrete time channels with inter-symbol interference (ISI) in the presence of channel uncertainties is investigated. The turbo equalization framework proposed in this paper contains a linear equalizer (LE) and a trellis based decoder. In this framework, a minimax scheme and a competitive scheme are studied, which incorporate the uncertainty in channel information into equalizer design in order to improve robustness. The validation of the performance improvement gained by the proposed algorithms are demonstrated through simulations.**

***Index Terms*— Channel uncertainties, Competitive, Linear Turbo Equalization, Minimax**

## I. Introduction

We consider robust turbo equalization over the frequency selective channels in the presence of channel uncertainties. Analogous to turbo decoding, turbo equalization follows a similar procedure as in classical turbo decoding procedure for the turbo codes with a replacement of the intentional error-correcting codes by the "unintentional" convolutional channel [1]. Since the parameters of this unintentional code are to be estimated by the receiver these "code parameters" are prone to estimation errors. The inaccuracies in the channel parameters may be either due to imperfect channel estimation or due to the time variations of the channel parameters outside the training period. In this article, we consider the design of novel turbo equalization approaches for achieving robustness against such potential uncertainties in the estimated channel parameters.

Turbo equalization systems were first studied in [2], where the received data is first processed by a maximum a posteriori (MAP) equalizer to combat the ISI and then the equalized data is decoded to obtain uncoded bits. The framework we investigate here, where the MAP equalizer is replaced by an LE is initially introuduced in [3], where a least mean square algorithm is used to obtain the LE coefficients. In [4] and [1] different extensions of this idea are further developed. Here, the LE coefficients are obtained by the Mean Square Error (MSE) minimization framework, where the channel inaccuracies are employed in the problem formulation for abstaining from completely tuning the LE parameters to the available inaccurate channel information. In the first approach, we apply a minimax framework where the LE coefficients are selected by minimizing the MSE with respect to the worst possible channel around the inaccurate channel coefficients [5], [6]. However, the minimax approach may result in overly conservative solutions in certain applications due to its design. For that reason, we extend this framework and define a relative performance measure between the MSE of an LE and the MMSE of the linear MMSE equalizer calculated with the correct knowledge of the underlying channel [7], [8]. Here, we emphasize that this relative performance measure describes our regret in using an LE that is not the correct linear MMSE (which is not available). As in the minimax case, unlike [8] and [7], this competitive setup has a bias term and a convolutive structure that needs different formulation specific to the turbo equalization framework. The problem of obtaining of coefficients of LEs for both approaches can be formulated as a semi-definite programming (SDP) problem, which can be efficiently solved [9].

The article is organized as follows. In Section II, the setup for turbo equalization is briefly described. In Section III, we illustrate the proposed equalization approaches. First, we consider the linear MMSE equalization tuned to the inaccurate channel coefficients. We then investigate the minimax approach and the competitive approach, and show that both problems can be cast as SDP problems. Section IV provides simulation results to demonstrate the performance of the proposed algorithms. Finally, we present the conclusions in Section V.

## II. Turbo Equalization System Description

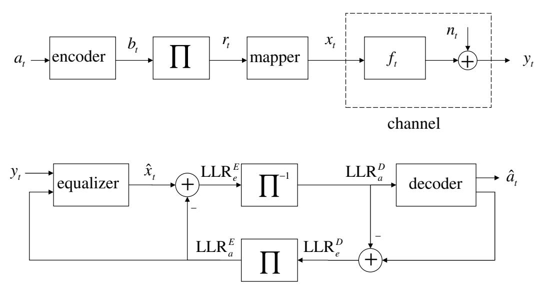


Fig. 1. A basic turbo equalization framework with the transmitter, the channel and the receiver. The receiver contains both the equalizer and the decoder.

The basic communication system studied in this paper is illustrated in Fig. 1. Here, $\{a_t\}$, is the transmitted signal. To incorporate redundancy in transmission, the

Throughout this paper, bold lowercase letters will denote vectors and bold uppercase letters will denote matrices. All vectors are column vectors and $l^2$-norm of a vector $\mathbf{v}$ is defined as $\|\mathbf{v}\| = \sqrt{\mathbf{v}^H\mathbf{v}}$, where $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^+$ represent transpose, conjugate transpose and conjugation, successively. The time index is shown in the subscripts. $\mathrm{E}[\cdot]$ denotes the expectation operator. For notational ease, the expected value of a random variable $x$ is denoted as $\bar{x} = \mathrm{E}[x]$, and the expected value of a random vector $\mathbf{x}$ is $\bar{\mathbf{x}} = \mathrm{E}[\mathbf{x}]$. $\mathbf{I}$ denotes the identity matrix, $\mathbf{0}$ represents a vector (or matrix) of zeros, where the dimensions are understood from the context. We denote by $\mathcal{N}(\mu, \sigma^2)$ the Gaussian distribution with mean $\mu$ and variance $\sigma^2$. "$*$" is the convolution operator.

input signal $\{a_t\}$ is encoded by a convolutional code to produce $\{b_t\}$. To further decrease the possible transmission errors, the encoded bits $\{b_t\}$ are interleaved using an S-random interleaver [10] to produce the interleaved bits $\{r_t\}$. Finally, the interleaved bits are modulated to produce the symbol sequence $\{x_t\}$. For notational simplicity in the sequel we use BPSK signaling i.e., $x_t = (-1)^{r_t+1}$. The modulated sequence $\{x_t\}$ is transmitted through a baseband discrete-time channel with a finite impulse response $\{f_t\}$, $t = 0, 1, \ldots, M-1$, represented by $\mathbf{f} \triangleq [f_{M-1}, \ldots, f_0]^T$. Here, the transmitted signal $\{x_t\}$ is assumed to be uncorrelated due to the interleaver. The received signal is given by $y_t \triangleq x_t * f_t + n_t$, where $\{n_t\}$ is the additive complex white Gaussian noise with zero mean and circular symmetric variance $\sigma_n^2$. Note that the underlying channel impulse response vector is not accurately known, however, an estimate of $\mathbf{f}$ is provided by $\tilde{\mathbf{f}}_t$ (which can be possibly time varying for certain adaptive methods [11]). The uncertainty in the channel impulse response vector is modeled by $\|\mathbf{f} - \tilde{\mathbf{f}}_t\| \leq \delta$, $\delta \in \mathbb{R}^+$, $\delta < \infty$, where $\delta$ or a bound on $\delta$ is known. Note that although the results we provide hold for time varying $\mathbf{f}_t$ and $\delta_t$ we have dropped the time index for notational ease.

In Fig. 1, the equalizer and decoder are considered as the inner decoder and outer decoder, respectively, and an iterative decoding scheme is used at the receiver. After $\{y_t\}$ is processed by a turbo equalization system (see Fig. 1), the equalizer computes the a posteriori information using the received signal, transmitted signal estimate, channel convolution matrix and *a priori* probability of the transmitted signals. Then by subtracting the *a priori* information $\mathrm{LLR}_a^E$ and by de-interleaving the extrinsic information $\mathrm{LLR}_e^E$, a soft input soft output channel decoder computes the extrinsic information $\mathrm{LLR}_e^D$ on coded bits, which are fed back to the LE as *a priori* information $\mathrm{LLR}_a^E$ after interleaving. The *a priori* information from the decoder can be used to compute the mean and variance of $x_t$ as $\bar{x}_t \triangleq E[x_t|\{\mathrm{LLR}_{a,t}^E\}]$ and $q_t \triangleq E[x_t^2|\{\mathrm{LLR}_{a,t}^E\}] - \bar{x}_t^2$, respectively. The mean and variance are given by $\bar{x}_t = \tanh(\mathrm{LLR}_{a,t}^E/2)$ and $q_t = 1 - |\bar{x}_t|^2$. We use an LE to mitigate the effect of ISI. The estimate of the desired data $x_t$ is modeled by

$$\hat{x}_t = \mathbf{c}_t^T \mathbf{y}_t + l_t + \bar{x}_t, \tag{1}$$

where $\mathbf{c}_t = [c_{t,N_2}, \ldots, c_{t,-N_1}]^T$ is length $N = N_1 + N_2 + 1$ LE and $\mathbf{y}_t \triangleq [y_{t-N_2}, \ldots, y_{t+N_1}]^T$. We underline that in (1) the equalizer is "affine" i.e., there is a bias term $l_t$ since the received signal $y_t$ is not zero mean and the mean sequence $\{\bar{y}_t\}$ is not known due to uncertainty in the channel. The received data vector is given by $\mathbf{y}_t = \mathbf{F}\mathbf{x}_t + \mathbf{n}_t$, where $\mathbf{x}_t \triangleq [x_{t-M-N_2+1}, \ldots, x_{t+N_1}]^T$ and $\mathbf{F} \in \mathbb{C}^{N \times (N+M-1)}$.

$$\mathbf{F} \triangleq \begin{bmatrix} f_{M-1} & \ldots & f_0 & 0 & \ldots & & 0 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & & 0 & f_{M-1} & \ldots & f_0 & \end{bmatrix}$$

is the convolution matrix constructed by $\mathbf{f} = [f_{M-1}, \ldots, f_0]^T$. The estimate of $x_t$, $\hat{x}_t$, can be written as

$$\hat{x}_t = \mathbf{c}_t^T \mathbf{F}\mathbf{x}_t + \mathbf{c}_t^T \mathbf{n}_t + l_t + \bar{x}_t. \tag{2}$$

Here $\mathbf{C}_t \in \mathbb{C}^{M \times (N+M-1)}$ is the convolution matrix of $\mathbf{c}_t$.

It is easy to show that for linear MMSE equalizer

$$\begin{aligned} \mathbf{c}_t &= [\mathbf{v}^T \mathbf{Q}_t \mathbf{F}^H (\sigma_n^2 \mathbf{I} + \mathbf{F}\mathbf{Q}_t\mathbf{F}^H)^{-1}]^T, \\ l_t &= \mathbf{c}_t^T \mathbf{F}\bar{\mathbf{x}}_t, \end{aligned} \tag{3}$$

where $\bar{\mathbf{x}}_t = [\bar{x}_{t-M-N_2+1}, \ldots, \bar{x}_{t+N_1}]^T$, and $\mathbf{Q}_t \triangleq E[(\mathbf{x}_t - \bar{\mathbf{x}}_t)(\mathbf{x}_t - \bar{\mathbf{x}}_t)^H]$ is a diagonal matrix (due to uncorrelateness assumption on $x_t$) with diagonal entries $\mathbf{Q}_t = \mathrm{diag}\left([q_{t-M-N_2+1}, \ldots, q_t, \ldots, q_{t+N_1}]\right)$ and $\mathbf{v} \in \mathbb{R}^{N+M-1}$ is a vector of all zeros except the $(M+N_2)$th entry is 1. Then, the corresponding linear MMSE is given by

$$\min_{\mathbf{c},l} E[\|x_t - \hat{x}_t\|^2] = \mathbf{v}^T (\mathbf{Q}_t^{-1} + \sigma_n^{-2}\mathbf{F}^H\mathbf{F})^{-1}\mathbf{v}. \tag{4}$$

To remove dependency of $\hat{x}_t$ on $\mathrm{LLR}_{a,t}^E$ due to using $\bar{x}_t$ and $q_t$ in (3) and (2), we set $\mathrm{LLR}_{a,t}^E$ to 0, yielding $\bar{x}_t = 0$ and $q_t = 1$ [1]. This changes the covariance matrix to $\mathbf{Q}'_t \triangleq \mathbf{Q} + (1 - q_t)\mathbf{v}\mathbf{v}^T$ and the mean vector $\bar{\mathbf{x}}_t$ to $\bar{\mathbf{x}}_t - \bar{x}_t\mathbf{v}$, resulting (2) and (3)

$$\begin{aligned} \mathbf{c}_t &= [\sigma_n^{-2}\mathbf{v}^T ({\mathbf{Q}'_t}^{-1} + \sigma_n^{-2}\mathbf{F}^H\mathbf{F})^{-1}\mathbf{F}^H]^T, \\ l_t &= \mathbf{c}_t^T \mathbf{F}(\bar{\mathbf{x}}_t - \bar{x}_t\mathbf{v}) \end{aligned} \tag{5}$$

$$\hat{x}_t = \mathbf{c}_t^T \mathbf{y}_t + l_t. \tag{6}$$

(5) or (6) can not be directly computed because the underlying channel vector $\mathbf{f}$ is not precisely known at the receiver, but an estimate $\tilde{\mathbf{f}}_t$, $\|\mathbf{f} - \tilde{\mathbf{f}}_t\| \leq \delta$, $\delta \in \mathbb{R}^+$, $\delta < \infty$ is provided.

## III. Equalization Methods

### A. Linear MMSE Equalization

When the channel $\mathbf{f}$ is not accurately known but estimated as $\tilde{\mathbf{f}}_t$, one may use a linear MMSE equalizer that is matched to the estimated channel vector $\tilde{\mathbf{f}}_t$ as

$$\begin{aligned} \tilde{\mathbf{c}}_t &= [\mathbf{v}^T \mathbf{Q}'_t \widetilde{\mathbf{F}}_t^H (\sigma_n^2 \mathbf{I} + \widetilde{\mathbf{F}}_t \mathbf{Q}'_t \widetilde{\mathbf{F}}_t^H)^{-1}]^T, \\ \tilde{l}_t &= \tilde{\mathbf{c}}_t^T \widetilde{\mathbf{F}}_t \bar{\mathbf{x}}_t, \end{aligned} \tag{7}$$

where $\widetilde{\mathbf{F}}_t$ is the convolution matrix generated by $\tilde{\mathbf{f}}_t$.

### B. Linear Equalization with A Minimax Formulation

When the channel $\mathbf{f}$ is unknown but estimated by $\tilde{\mathbf{f}}_t$, one can incorporate the uncertainty in the channel estimate in the equalizer design using a minimax framework in order to improve robustness over (7). In this minimax framework, the equalizer coefficients are obtained by minimizing the MSE with respect to the worst possible channel around the channel estimate $\tilde{\mathbf{f}}_t$, i.e.,

$$\{\tilde{\mathbf{c}}_t^{\mathrm{MM}}, \tilde{l}_t^{\mathrm{MM}}\} = \tag{8}$$

$$\arg\min_{\mathbf{c},l} \max_{\mathbf{f}=\tilde{\mathbf{f}}_t+\mathbf{df}, \|\mathbf{df}\| \leq \delta} \left\{ E[|x_t - \bar{x}_t - \mathbf{c}^T\mathbf{y}_t - l|^2] \right\}.$$

Note that

$$E[|x_t - \bar{x}_t - \mathbf{c}^T\mathbf{y}_t - l|^2] \quad (9)$$
$$= (\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2,$$

where $\mathbf{C}$ is the convolution matrix of $\mathbf{c}$. (9) follows since $n_t$ is i.i.d. and $(\mathbf{x}_t - \bar{\mathbf{x}}_t)$ has zero mean. The following theorem, proved in [12], shows that the minimax equalizer coefficients $\{\tilde{\mathbf{c}}_t^{\mathrm{MM}}, \tilde{l}_t^{\mathrm{MM}}\}$ satisfying (8) is the solution of an SDP problem.

**Theorem 1:** Let $\{x_t\}$, $\{y_t\}$ and $\{n_t\}$ represent the transmitted, received and noise sequences in Fig. 1 such that $y_t = f_t * x_t + n_t$, where $\mathbf{f} = [f_{M-1}, \ldots, f_0]^T$ is the unknown, possibly time varying, channel impulse response vector and $n_t$ is zero mean. At each time $t$, given an estimate $\tilde{\mathbf{f}}_t$ of the underlying communication channel response vector $\mathbf{f}$ satisfying $\mathbf{f} = \tilde{\mathbf{f}}_t + \mathbf{df}$,$\|\mathbf{df}\| \le \delta$, then the problem

$$\underset{\mathbf{c},l}{\text{minimize}} \underset{\mathbf{f}=\tilde{\mathbf{f}}_t+\mathbf{df},\|\mathbf{df}\|\le\delta}{\text{maximize}} [(\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2], \quad (10)$$

where $\mathbf{c} = [c_{N_2}, \ldots, c_{-N_1}]^T$ and $l$ are the coefficients of the LE, $\mathbf{C}$ is the convolution matrix generated from $\mathbf{c}$, $\mathbf{Q}'_t = \mathbf{Q}_t - (1 - q_t)\mathbf{v}\mathbf{v}^T$ and $E[\mathbf{n}_t\mathbf{n}_t^H] = \sigma_n^2\mathbf{I}$ are the covariance matrices of the transmitted and noise sequences, respectively, is equivalent to the SDP problem

$$\underset{\alpha,\mathbf{c},l,\tau}{\text{minimize}}\, \alpha \quad (11)$$

such that

$$\begin{bmatrix} \alpha - \tau & \mathbf{c}^H & \mathbf{w}^H & \mathbf{s} & \mathbf{0} \\ \mathbf{c} & \sigma_n^{-2}\mathbf{I} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{w} & \mathbf{0} & \mathbf{Q}'^{-1}_t & \mathbf{0} & -\delta\mathbf{C}^T \\ \mathbf{s}^H & \mathbf{0} & \mathbf{0} & 1 & \delta\bar{\mathbf{x}}_t^T\mathbf{C}^T \\ \mathbf{0} & \mathbf{0} & -\delta\mathbf{C}^+ & \delta\mathbf{C}^+\bar{\mathbf{x}}_t^+ & \tau\mathbf{I} \end{bmatrix} \ge 0,$$

where $\mathbf{s} = (l + \bar{\mathbf{x}}_t^T\mathbf{C}^T\tilde{\mathbf{f}}_t)^H$ and $\mathbf{w} = \mathbf{v} - \mathbf{C}^T\tilde{\mathbf{f}}_t$. The minimizer $\{\mathbf{c}, l\}$ in (11) yields the robust linear equalizer $\{\tilde{\mathbf{c}}_t^{\mathrm{MM}}, \tilde{l}_t^{\mathrm{MM}}\}$ in (8).

### *C. Linear Equalization With Competitive Algorithm Formulation*

In the minimax framework, the LE coefficients are obtained by minimizing the MSE corresponding to the worst possible channel around the channel estimate. This may result in overly conservative solutions in certain applications. Therefore, following [7], [8], we propose a competitive approach to improve the equalization performance, while trying to preserve robustness. As opposed to the minimax approach, instead of the MSE performance, the performance of an LE is defined with respect to the MMSE LE tuned to the underlying unknown channel. For any affine equalizer coefficients $\{\mathbf{c}, l\}$, we define our regret for using the LE coefficients $\{\mathbf{c}, l\}$ instead of the MMSE LE tuned to $\mathbf{f}$ as

$$E[|x_t - \bar{x}_t - \mathbf{c}^T\mathbf{y}_t - l|^2] - \left(\min_{\mathbf{w},r} E[|x_t - \bar{x}_t - \mathbf{w}^T\mathbf{y}_t - r)|^2]\right)$$
$$= \left[(\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2\right] - \left(\mathbf{v}^T[\mathbf{Q}'^{-1}_t + \sigma_n^{-2}\mathbf{F}^H\mathbf{F}]^{-1}\mathbf{v}\right), \quad (12)$$

where (9) and (4) are used in (12). However, to make the SDP problem formulation tractable, instead of directly using $(\mathbf{v}^T[\mathbf{Q}'^{-1}_t + \sigma_n^{-2}\mathbf{F}^H\mathbf{F}]^{-1}\mathbf{v})$ in the regret formulation of (12), one can use a first order Taylor approximation around $\tilde{\mathbf{f}}_t$ [8], as

$$\mathbf{v}^T[\mathbf{Q}'^{-1}_t + \sigma_n^{-2}\mathbf{F}^H\mathbf{F}]^{-1}\mathbf{v} = \eta_t + \mathbf{df}^H\mathbf{g}_t{}^+ + \mathbf{g}_t{}^T\mathbf{df} + O(\|\mathbf{df}\|^2),$$

where $\eta_t \triangleq \mathbf{v}^T[\mathbf{Q}'^{-1}_t + \sigma_n^{-2}\widetilde{\mathbf{F}}_t^H\widetilde{\mathbf{F}}_t]^{-1}\mathbf{v}$, $\mathbf{g}_t \triangleq -\widetilde{\mathbf{C}}_t(\mathbf{Q}'_t + \sigma_n^{-2}\widetilde{\mathbf{F}}_t^H\widetilde{\mathbf{F}}_t)^{-1}\mathbf{v}$ and $\widetilde{\mathbf{C}}_t$ is the convolution matrix constructed using $\tilde{\mathbf{c}}_t$ in (7). Employing this in (12) yields the regret as

$$\left[(\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2\right] - \left(\eta_t + \mathbf{df}^H\mathbf{g}_t{}^+ + \mathbf{g}_t{}^T\mathbf{df}\right), \quad (13)$$

where the $O(\|\mathbf{df}\|^2)$ term is left out. The effect of this approximation diminishes as $\|\mathbf{df}\|$ gets smaller. For distortions with larger $\|\mathbf{df}\|$, one can use higher order Taylor approximations instead. However, we have observed through our simulations that the solution using the first order approximation yields satisfactory results even for fairly large $\|\mathbf{df}\|$ (when compared to $\|\mathbf{f}\|$). To get the competitive LE, we minimize this regret over all possible channels around the channel estimate, $\tilde{\mathbf{f}}_t$, i.e.,

$$\{\tilde{\mathbf{c}}_t^{\mathrm{CP}}, \tilde{l}_t^{\mathrm{CP}}\} \quad (14)$$
$$= \arg\min_{\mathbf{c},l} \max_{\mathbf{f}=\tilde{\mathbf{f}}_t+\mathbf{df},\|\mathbf{df}\|\le\delta} [(\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2 - (\eta_t + \mathbf{df}^H\mathbf{g}_t{}^+ + \mathbf{g}_t{}^T\mathbf{df})].$$

The problem in (14) whose solution will yield the corresponding competitive LE can be formulated as an SDP problem.

**Theorem 2:** Let $\{x_t\}$, $\{y_t\}$ and $\{n_t\}$ represent the transmitted, received and noise sequences in Fig. 1 such that $y_t = f_t * x_t + n_t$, where $\mathbf{f}$ is the unknown, possibly time varying, channel impulse response vector and $n_t$ is zero mean. At each time $t$, given an estimate $\tilde{\mathbf{f}}_t$ of the underlying communication channel impulse response vector $\mathbf{f}$ satisfying $\mathbf{f} = \tilde{\mathbf{f}}_t + \mathbf{df}$, $\|\mathbf{df}\| \le \delta$, then the problem

$$\underset{\mathbf{c},l}{\text{minimize}} \underset{\mathbf{f}=\tilde{\mathbf{f}}_t+\mathbf{df},\|\mathbf{df}\|\le\delta}{\text{maximize}} \Big[(\mathbf{v} - \mathbf{C}^T\mathbf{f})^H\mathbf{Q}'_t(\mathbf{v} - \mathbf{C}^T\mathbf{f}) + \sigma_n^2\mathbf{c}^H\mathbf{c} + |l + \mathbf{f}^T\mathbf{C}\bar{\mathbf{x}}_t|^2 - (\eta_t + \mathbf{df}^H\mathbf{g}_t{}^+ + \mathbf{g}_t{}^T\mathbf{df})\Big] \quad (15)$$

where $\mathbf{c} = [c_{N_2}, \ldots, c_{-N_1}]^T$ and $l$ are the coefficients of the linear equalizer, $\mathbf{C}$ is the convolution matrix generated from $\mathbf{c}$, $\mathbf{Q}'_t = \mathbf{Q}_t - (1 - q_t)\mathbf{v}\mathbf{v}^T$ and $E[\mathbf{n}_t\mathbf{n}_t^H] = \sigma_n^2\mathbf{I}$ are the covariance matrices of the transmitted and noise sequences, respectively, $\eta_t = \mathbf{v}^T[\mathbf{Q}'^{-1}_t + \sigma_n^{-2}\widetilde{\mathbf{F}}_t^H\widetilde{\mathbf{F}}_t]^{-1}\mathbf{v}$, $\mathbf{g}_t = -\widetilde{\mathbf{C}}_t(\mathbf{Q}'_t + \sigma_n^{-2}\widetilde{\mathbf{F}}_t^H\widetilde{\mathbf{F}}_t)^{-1}\mathbf{v} > 0$, is equivalent to the SDP problem

$$\underset{\alpha,\mathbf{c},l,\tau}{\text{minimize}}\, \alpha \quad (16)$$

such that

$$\begin{bmatrix} \alpha+\eta_t-\tau & \mathbf{c}^H & \mathbf{w}^H & \mathbf{s} & \delta\mathbf{g}_t{}^T \\ \mathbf{c} & \sigma_n^{-2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{w} & \mathbf{0} & \mathbf{Q}_t^{'-1} & \mathbf{0} & -\delta\mathbf{C}^T \\ \mathbf{s}^H & \mathbf{0} & \mathbf{0} & 1 & \delta\bar{\mathbf{x}}_t^T\mathbf{C}^T \\ \delta\mathbf{g}_t{}^+ & \mathbf{0} & -\delta\mathbf{C}^+ & \delta\mathbf{C}^+\bar{\mathbf{x}}_t^+ & \tau\mathbf{I} \end{bmatrix} \geq 0, \tag{17}$$

where $\mathbf{s} = (l + \bar{\mathbf{x}}_t^T\mathbf{C}^T\tilde{\mathbf{f}}_t)^H$ and $\mathbf{w} = \mathbf{v} - \mathbf{C}^T\tilde{\mathbf{f}}_t$. The minimizer $\{\mathbf{c}, l\}$ in (16) is the competitive LE coefficients $\{\tilde{\mathbf{c}}_t^{\mathrm{CP}}, \tilde{l}_t^{\mathrm{CP}}\}$ in (14). The proof of the theorem is provided in [12].

## IV. Simulations

In this section, we compare the performances of the proposed algorithms with the performance of the plug-in MMSE equalization algorithm in turbo equalization framework. We use the simulation setup from [13]. Here, we employ a convolutional encoder with a generator matrix $G = \begin{bmatrix} 1 & 0 & D^2; 1 & D & D^2 \end{bmatrix}$ [13] to encode the transmitted bits. To shuffle the coded bits of length 1024 we use an 8-random interleaver [10]. The coded bits are then BPSK modulated. We use LEs introduced in the text and a MAP based algorithm for decoding [13], [14].

In the experiments, the modulated bits are transmitted through the ISI channel from [14] (Chapter 10) $\mathbf{f} = [0.227 \quad 0.46 \quad 0.688 \quad 0.46 \quad 0.227]^T$, with $||\mathbf{f}|| = 1$, $M = 5$ and the noise variance $\sigma_n^2$ is determined by $\mathrm{SNR} = E[||x_t||^2]/N_0 = 1/2\sigma_n^2$. The channel estimates are constructed using $\tilde{\mathbf{f}} = \mathbf{f} + \mathbf{df}$, where the distortion $\mathbf{df}$ is randomly generated using a standard Normal distribution, $\mathcal{N}(0, 1)$. In the experiment, we scale the bound for the norm of $\mathbf{df}$ inversely proportional to SNR since the channel estimates usually deteriorate with low SNR [14]. Based on some empirical data we assume that $\|\mathbf{df}\| \leq 0.4 - 0.25\,\mathrm{SNR}/6$. The length of LE is $N = 15$, where $N_1 = 7$ (non-causal part) and $N_2 = 7$ (causal part). In simulations, at each SNR, BERs are averaged over 200 random $\mathbf{df}$s. In Fig. 2, we present average BERs corresponding to the LEs $\tilde{\mathbf{c}}_t$ from (7) "mmse", $\tilde{\mathbf{c}}_t^{\mathrm{MM}}$ from (8) "minimax" and $\tilde{\mathbf{c}}_t^{\mathrm{CP}}$ from (14) "regret". Since the LE based on MMSE criterion closely approximates the optimal LE which directly minimizes BER we expect the BER values of minimax algorithm to be the lowest among other algorithms for the "worst channels". Although the proposed algorithms are optimized with respect to the worst case MSE or to the worst case regret, their average BER performance is close to each other for low SNRs, and for high SNRs, with the increase in iteration count the average BER performance of the proposed algorithms outperform the performance of the plug-in MMSE equalization algorithm.

## V. Conclusion

We proposed two novel approaches in the linear turbo equalization context under the constraint that the underlying discrete time communication channel is not accurately known. A minimax and a competetive approaches are studied to incorporate the uncertainty in the channel coefficients into the problem formulation to mitigate the effect of the uncertainties in the channel parameters. Through simulations we observed that the introduced methods improve over the plug-in MMSE method under different distortions and SNRs. The performance gains of the proposed algorithms become more evident as the iteration number increases.

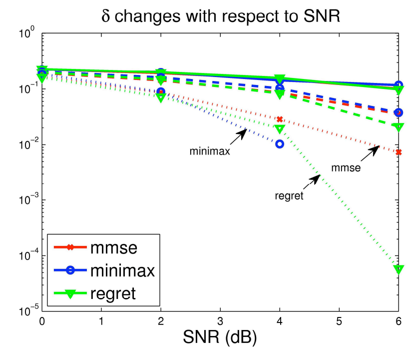


Fig. 2. Average BERs under different SNRs. Here, the first iteration (the straight lines), the second iteration (dashed lines) and the fourth iteration (the dotted lines).

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