PROBLEM 1 (20 points): Prove or disprove: A continuous function \( f : (0, 1) \rightarrow \mathbb{R} \) is bounded.

**ANSWER:** False.
The function \( f : (0, 1) \rightarrow \mathbb{R}, \ f(x) = \frac{1}{x} \) which is continuous, but not bounded.

PROBLEM 2 (20 points): Prove or disprove: A **uniformly** continuous function \( f : (0, 1) \rightarrow \mathbb{R} \) is bounded.

**ANSWER:** True.
Take \( \epsilon = 1 \), then as \( f \) is uniformly continuous, \( \exists \delta > 0 \) such that \( \forall x, y \in (0, 1) \ |x - y| < \delta \) implies \( |f(x) - f(y)| < 1 \). Now take some distinct points in (0, 1) such that the distance between any two consecutive points is less than \( \delta \). For example, take \( n \in \mathbb{N} \) such that \( 1/n < \delta \), and let \( a_i = i/n, 0 < i < n \). Then for an arbitrary \( x \in (0, 1) \), we have \( |x - a_k| < \delta \) for some \( k \). This implies that \( |f(x) - f(a_k)| < 1 \) by uniform continuity. Let \( M = \max\{|f(a_i)|, 0 < i < n\} \). Thus we get \( |f(x)| < 1 + M \) which certainly implies that \( f \) is bounded.

PROBLEM 3 (20 points): If \( f : [-1, 1] \rightarrow \mathbb{R} \) is continuous, \( f(-1) > -1 \) and \( f(1) < 1 \), show that there exists a point \( c \in (-1, 1) \) such that \( f(c) = c \).

**ANSWER:**
Let \( h : [-1, 1] \rightarrow \mathbb{R} \) be a function defined as \( h(x) = f(x) - x \) \( \Rightarrow \) \( h \) is continuous, \( h(-1) = f(-1) - (-1) > 0 \), \( h(1) = f(1) - 1 < 0 \). So by the Intermediate Value Theorem, \( \exists c \in (-1, 1) \) such that \( h(c) = f(c) - c = 0 \).

PROBLEM 4 (20 points): For a function \( f : D \rightarrow \mathbb{R} \) and \( x_0 \) in \( D \), define \( A = \{ x \in D | x \geq x_0 \} \) and \( B = \{ x \in D | x \leq x_0 \} \). Prove that if \( f : A \rightarrow \mathbb{R} \) and \( f : B \rightarrow \mathbb{R} \) are continuous at \( x_0 \), then \( f : D \rightarrow \mathbb{R} \) is continuous at \( x_0 \).

**ANSWER:**
Let \( \epsilon > 0 \) be given. Then \( \exists \delta_1 > 0 \) such that \( |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \epsilon \), for all \( x \in A \) and \( \exists \delta_2 > 0 \) such that \( |x - x_0| < \delta_2 \Rightarrow |f(x) - f(x_0)| < \epsilon \), for all \( x \in B \). Setting \( \delta = \min\{\delta_1, \delta_2\} \) we get \( |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \) for all \( x \in D \), i.e., \( f \) is continuous at \( x_0 \).

PROBLEM 5 (20 points): Prove or disprove: The set of irrationals is closed in \( \mathbb{R} \).

**ANSWER:** False.
We know that \( \mathbb{R} \setminus \mathbb{Q} \) is dense in \( \mathbb{R} \). This means that every number is limit of a sequence in \( \mathbb{R} \setminus \mathbb{Q} \). Therefore, for \( q \in \mathbb{Q}, \exists \{x_n\} \in \mathbb{R} \setminus \mathbb{Q} \) such that \( \lim_{n \to \infty} x_n = q \) and thus \( \mathbb{R} \setminus \mathbb{Q} \) is not closed.

PROBLEM 6 (20 points): Suppose that the function \( g : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and that \( g(x) = \sqrt{e} \) for every rational number \( x \). What is \( g(\sqrt{2}) \)? Prove your answer.

**ANSWER:** \( g(\sqrt{2}) = \sqrt{e} \)
We know that there exists a rational sequence \( \{x_n\} \) converging to \( \sqrt{2} \), and \( g \) is continuous. So \( g(\sqrt{2}) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} \sqrt{e} = \sqrt{e} \).