KOÇ UNIVERSITY, SPRING 2018, MATH 208, QUIZ-1, FEBRUARY 19

PROBLEM 1 (15 points): A real number of the form \( \frac{m}{2^n} \), where \( m, n \in \mathbb{Z} \), is called a dyadic rational. Prove that the set of dyadic rationals is dense in \( \mathbb{R} \).

SOLUTION:
Let \( a \) and \( b \) be two real numbers such that \( a < b \). By the Archimedean Property, there exists \( n \in \mathbb{N} \) such that \( 0 < \frac{1}{n} < b - a \), which implies \( 0 < \frac{1}{2^n} < \frac{1}{n} < b - a \).

Thus we have \( 1 < b2^n - a2^n \). As the distance between \( b2^n \) and \( a2^n \) is greater than 1, there exists \( m \in \mathbb{N} \) such that \( a2^n < m < b2^n \) which implies that \( a < \frac{m}{2^n} < b \). So we proved that for each open interval \( (a, b) \subset \mathbb{R} \), there exists a rational number of the form \( \frac{m}{2^n} \) which belongs to \( (a, b) \). In other words, the set of dyadic rationals is dense in \( \mathbb{R} \).

PROBLEM 2 (15 points): Suppose that \( \{a_n\} \) is a monotonically decreasing sequence. Show that if \( \{a_n\} \) has a bounded subsequence, then \( \{a_n\} \) is bounded.

SOLUTION:
Assume that \( \{a_n\} \) is a monotonically decreasing sequence which has a bounded subsequence \( \{a_{n_k}\} \). Since \( \{a_n\} \) is monotonically decreasing, we know that \( a_1 \) is an upper bound. Therefore, to solve the given problem, it suffices to show that \( \{a_n\} \) is also bounded below. Now, since \( \{a_{n_k}\} \) is bounded, it is bounded below which means that there exists \( M \in \mathbb{R} \) such that \( M \leq a_{n_k} \) for any \( k \in \mathbb{N} \). But then for each \( k \in \mathbb{N} \), we have \( k \leq n_k \), which implies that \( M \leq a_{n_k} \leq a_k \), since \( \{a_n\} \) is a monotonically decreasing sequence. We conclude that \( M \leq a_k \), for all \( k \in \mathbb{N} \). Thus the sequence \( \{a_n\} \) is also bounded below and hence bounded.

PROBLEM 3 (20 points): Consider the quadratic equation \( x^2 - x - 1 = 0, \ x > 0 \). Define the sequence \( \{x_n\} \) recursively by fixing \( x_1 \) and then defining \( x_{n+1} = \sqrt{1 + x_n} \) for \( n \in \mathbb{N} \). Prove that the sequence \( \{x_n\} \) converges monotonically to the solution of the above equation.

SOLUTION:
There are two roots of the equation \( x^2 - x - 1 = 0 \), which are \( \frac{1 \pm \sqrt{5}}{2} \). The only positive root is \( \frac{1 + \sqrt{5}}{2} \).

Let \( x_1 > 0 \). If \( x_1 = \frac{1 + \sqrt{5}}{2} \), then

\[
x_2 = \sqrt{1 + x_1} = \sqrt{1 + \frac{1 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2}.
\]

Observe that, if \( x_k = \frac{1 + \sqrt{5}}{2} \), then

\[
x_{k+1} = \sqrt{1 + x_k} = \sqrt{1 + \frac{1 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2},
\]

for all \( k \).
and thus \( \{x_n\} \) is the constant sequence converging to \( \frac{1 + \sqrt{5}}{2} \), by induction.

Next, we consider the nontrivial cases where \( 0 < x_1 < \frac{1 + \sqrt{5}}{2} \) or \( x_1 > \frac{1 + \sqrt{5}}{2} \).

**Case 1:** Suppose that \( 0 < x_1 < \frac{1 + \sqrt{5}}{2} \). This implies that \( x_1^2 - x_1 - 1 = (x_1 - \frac{1 + \sqrt{5}}{2})(x_1 - \frac{1 - \sqrt{5}}{2}) < 0 \).

Then we claim that \( \{x_n\} \) is a monotonically increasing sequence converging to \( \frac{1 + \sqrt{5}}{2} \). To see this, we first show that \( x_n < \frac{1 + \sqrt{5}}{2} \) for all \( n \in \mathbb{N} \) by induction as follows: first of all \( x_1 < \frac{1 + \sqrt{5}}{2} \) and assuming \( x_k < \frac{1 + \sqrt{5}}{2} \) for some \( k \in \mathbb{N} \), we get

\[
x_{k+1} = \sqrt{1 + x_k} < \sqrt{1 + \frac{1 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2}.
\]

Next we claim that \( x_{n+1} > x_n \) for all \( n \in \mathbb{N} \). To show this,

\[
x_{n+1}^2 = (\sqrt{1 + x_n})^2 = 1 + x_n > x_n^2
\]

where the last inequality is equivalent to \( x_n^2 - x_n - 1 < 0 \), which is true since we proved that \( x_n < \frac{1 + \sqrt{5}}{2} \) for all \( n \in \mathbb{N} \).

Hence \( \{x_n\} \) is bounded above by \( \frac{1 + \sqrt{5}}{2} \) and monotonically increasing. Therefore, by the monotone convergence theorem, \( \{x_n\} \) converges to some \( L \in \mathbb{R} \). Note that \( \{x_{n+1}\} \) (a subsequence of \( \{x_n\} \)) converges to \( L \) as well and thus \( L = \sqrt{1 + L} \). Although there are two solutions of this equation, we conclude that \( L = \frac{1 + \sqrt{5}}{2} \), since a sequence of positive terms cannot converge to a negative number. We conclude that \( \{x_n\} \) is a monotonically increasing sequence converging to the positive solution \( \frac{1 + \sqrt{5}}{2} \) of the above equation.

**Case 2:** Suppose that \( x_1 > \frac{1 + \sqrt{5}}{2} \). This implies that \( x_1^2 - x_1 - 1 = (x_1 - \frac{1 + \sqrt{5}}{2})(x_1 - \frac{1 - \sqrt{5}}{2}) > 0 \).

Then we claim that \( \{x_n\} \) is a monotonically decreasing sequence converging to \( \frac{1 + \sqrt{5}}{2} \). To see this, we first show that \( x_n > \frac{1 + \sqrt{5}}{2} \) for all \( n \in \mathbb{N} \) by induction as follows: first of all \( x_1 > \frac{1 + \sqrt{5}}{2} \) and assuming \( x_k > \frac{1 + \sqrt{5}}{2} \) for some \( k \in \mathbb{N} \), we get

\[
x_{k+1} = \sqrt{1 + x_k} > \sqrt{1 + \frac{1 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2}.
\]

Next we claim that \( x_{n+1} < x_n \) for all \( n \in \mathbb{N} \). To show this,

\[
x_{n+1}^2 = (\sqrt{1 + x_n})^2 = 1 + x_n < x_n^2
\]
where the last inequality is equivalent to \( x_n^2 - x_n - 1 > 0 \), which is true since we proved that \( x_n > \frac{1 + \sqrt{5}}{2} \) for all \( n \in \mathbb{N} \).

Hence \( \{x_n\} \) is bounded below by \( \frac{1 + \sqrt{5}}{2} \) and monotonically decreasing. Therefore, by the monotone convergence theorem, \( \{x_n\} \) converges to some \( L \in \mathbb{R} \). Note that \( \{x_{n+1}\} \) (a subsequence of \( \{x_n\} \)) converges to \( L \) as well and thus \( L = \sqrt{1 + L} \). Although there are two solutions of this equation, we conclude that \( L = \frac{1 + \sqrt{5}}{2} \), since a sequence of positive terms cannot converge to a negative number. We conclude that \( \{x_n\} \) is a monotonically decreasing sequence converging to the positive solution \( \frac{1 + \sqrt{5}}{2} \) of the above equation.