PROBLEM 1 (15 points): A real number of the form \( \frac{m}{2^n} \), where \( m, n \in \mathbb{Z} \), is called a dyadic rational. Prove that the set of dyadic rationals is dense in \( \mathbb{R} \).

**ANSWER:**
We take \( a < b \in \mathbb{R} \). By Archimedean Property \( \exists n \in \mathbb{N} \) such that
\[
0 < \frac{1}{n} < b - a,
\]
which implies \( 0 < \frac{1}{2^n} < \frac{1}{n} < b - a \).

Thus we have \( 1 < 2^n \cdot b - 2^n \cdot a \). As the distance between \( 2^n \cdot b \) and \( 2^n \cdot a \) is greater than 1, there is an integer \( m \) such that \( 2^n \cdot a < m < 2^n \cdot b \Rightarrow a < \frac{m}{2^n} < b \), \( (2^n \neq 0) \). So the set of dyadic rationals is dense in \( \mathbb{R} \).

PROBLEM 2 (15 points): Show that a monotone sequence which has a bounded subsequence is bounded.

**ANSWER:**
Assume that \( \{x_n\} \) is an increasing sequence which has a bounded subsequence \( \{x_{n_k}\} \). Now, as \( \{x_{n_k}\} \) is bounded \( \exists M \in \mathbb{R} \) such that \( x_{n_k} \leq M \) for every element \( x_{n_k} \) of the sequence. But then for an element \( x_n \), \( \exists k \in \mathbb{N} \) such that \( n_k > n \) implying that \( x_n \leq x_{n_k} \leq M \), as \( \{x_n\} \) is increasing. Since \( n \) is arbitrary we have shown that \( x_n \leq M \), \( \forall n \in \mathbb{N} \). Therefore the sequence \( \{x_n\} \) is also bounded.

PROBLEM 3 (20 points): Consider the quadratic equation \( x^2 - x - 1 = 0 \), \( x > 0 \). Define the sequence \( \{x_n\} \) recursively by fixing \( x_1 \) and then defining \( x_{n+1} = \sqrt{1 + x_n} \) for \( n \in \mathbb{N} \). Prove that the sequence \( \{x_n\} \) converges monotonically to the solution of the above equation.

**ANSWER:**
Let \( x_1 > 0 \). We claim that if \( x_1^2 - x_1 - 1 < 0 \), then \( \{x_n\} \) is monotonically increasing sequence converging to \( \frac{1 + \sqrt{5}}{2} \): \( x_2 > x_1 \) since \( x_2^2 = (\sqrt{1 + x_1})^2 = 1 + x_1 > x_1^2 \) by assumption. By induction, \( x_{n+1} > x_n \).

Moreover \( x_n < \frac{1 + \sqrt{5}}{2} \) by induction again since \( x_{n+1} = \sqrt{1 + x_n} < \sqrt{1 + \frac{1 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2} \). Hence \( \{x_n\} \) is bounded above and monotone increasing \( \Rightarrow \lim_{n \to \infty} x_n = L \). Note that \( \{x_{n+1}\} \to L \) as well and, hence \( L = \sqrt{1 + L} \Rightarrow L = \frac{1 + \sqrt{5}}{2} \).

If \( x_1^2 - x_1 - 1 > 0 \), then \( \{x_n\} \) is monotonically decreasing sequence converging to \( \frac{1 + \sqrt{5}}{2} \) and similar to above, \( \{x_n\} \) converges to its infimum : \( \frac{1 + \sqrt{5}}{2} \).