PROBLEM 1 (25 points): Show that there does not exist a strictly increasing function \( f : \mathbb{Q} \to \mathbb{R} \) such that \( f(\mathbb{Q}) = \mathbb{R} \).

**ANSWER**

"Proof. Suppose that \( f : \mathbb{Q} \to \mathbb{R} \) is a strictly increasing function such that \( f(\mathbb{Q}) = \mathbb{R} \). This would imply that \( f \) is 1-1 and onto, which is impossible since \( \mathbb{Q} \) is countable and \( \mathbb{R} \) is uncountable. \( \square \)"

PROBLEM 2 (25 points): Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) has the property that there is some \( M > 0 \) such that \( |f(x)| \leq M|x|^2 \) for all \( x \). Prove that

\[
\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{f(x)}{x} = 0
\]

**ANSWER**

"Proof. Let \( \{x_n\} \) be a sequence in \( \mathbb{R} \setminus \{0\} \) such that \( \lim_{n \to \infty} x_n = 0 \). Thus we have
\[
|f(x_n)| \leq M|x_n|^2 \to 0,
\]

which implies that

\[
\lim_{n \to \infty} |f(x_n)| \leq \lim_{n \to \infty} M|x_n|^2 = 0 \Rightarrow \lim_{n \to \infty} |f(x_n)| = 0 \Rightarrow \lim_{n \to \infty} f(x_n) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0
\]

\[
\frac{|f(x_n)|}{|x_n|} \leq M|x_n| \Rightarrow \lim_{n \to \infty} \frac{|f(x_n)|}{|x_n|} \leq \lim_{n \to \infty} M|x_n| = 0 \Rightarrow \lim_{n \to \infty} \frac{f(x_n)}{x_n} = 0 \Rightarrow \lim_{x \to 0} \frac{f(x)}{x} = 0
\]

\( \square \)

PROBLEM 3 (25 points): For \( m_1, m_2 \in \mathbb{R} \), with \( m_1 \neq m_2 \), define

\[
f(x) = \begin{cases} 
m_1x + 4 & \text{if } x \leq 0 \\ 
m_2x + 4 & \text{if } x \geq 0. 
\end{cases}
\]

Prove that the function \( f : \mathbb{R} \to \mathbb{R} \) is continuous but not differentiable at \( x = 0 \).

**ANSWER**

"Proof. Since \( \lim_{x \to 0} f(x) \) exists and is equal to \( f(0) = 4 \), \( f \) is continuous at zero. But \( f \) is not differentiable at zero, since we have
\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{m_2x + 4 - 4}{x} = m_2 \neq m_1 = \lim_{x \to 0^-} \frac{m_1x + 4 - 4}{x} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}. 
\]"
PROBLEM 4 (25 points): Prove that if a function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and odd, \( f' : \mathbb{R} \to \mathbb{R} \) is even.

**ANSWER**

*Proof.* We are given that \( f \) is an odd function, so \( f(x) = -f(-x), \forall x \in \mathbb{R} \). Since \( f \) is differentiable, we may differentiate both sides of this equation:

\[
\frac{df(x)}{dx} = \frac{d(-f(-x))}{dx}
\]

\[
f'(x) = -(f'(-x)) \cdot (-1) \quad \text{(by the Chain Rule)}
\]

\[
f'(x) = f'(-x)
\]

Therefore \( f' \) is an even function.

\( \square \)