KOÇ UNIVERSITY, SPRING 2007
MATH 208 FINAL EXAM, May 25
Instructor: BURAK OZBAGCI
TIME: 3 Hours

PROBLEM 1 (10 points): Prove that every convergent sequence of real numbers is bounded.

PROBLEM 2 (10 points): Give an example of a sequence of real numbers with uncountably many cluster points. Prove your answer.

PROBLEM 3 (10 points): If \((x_p)\) is a convergent sequence in \(\mathbb{R}^5\) with \(x_p \to x\) then show that \(\|x_p\| \to \|x\|\).

PROBLEM 4 (10 points): Use the Intermediate Value Theorem to prove that every polynomial (of real numbers) of odd degree has at least one real root.

PROBLEM 5 (20 points):
   a) Let \(A = (2, 5] \cup \{1, 6\}\) and \(B = \{1, 5\} \cup [2, 3)\) be subsets of \(\mathbb{R}\). Let \(C = A \times B \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2\). Find the interior of \(C \subset \mathbb{R}^2\). Prove your answer.
   b) Is the set \(C \subset \mathbb{R}^2\) in Problem 5a compact? Prove your answer.

PROBLEM 6 (10 points): Decide whether the set \(D \subset \mathbb{R}^2\) below is connected in \(\mathbb{R}^2\) or not. Prove your answer.

\[
D = \{(x, 0) : 0 \neq x \in \mathbb{R}\} \cup \{(0, y) : 0 \neq y \in \mathbb{R}\}.
\]

PROBLEM 7 (10 points): Let \(f : \mathbb{R}^3 \to \mathbb{R}\) be a function such that

\[|f(x)| \leq \|x\|^2.\]

Show that \(f\) is differentiable at \(0 \in \mathbb{R}^3\) and find its differential.

PROBLEM 8 (10 points): Let \(F : \mathbb{R}^3 \to \mathbb{R}^7\) be the map defined as follows

\[F(x, y, z) = (2x, -\cos y, 3z^2, e^{xyz}, 0, \ln(1 + y^2), -xz^3)\]

Prove that \(F\) is differentiable on \(\mathbb{R}^3\) and find its derivative at \((x, y, z) \in \mathbb{R}^3\).

PROBLEM 9 (10 points): If \(f : \mathbb{R}^8 \to \mathbb{R}^9\) is differentiable at a point \(a \in \mathbb{R}^8\) then show that \(f\) is continuous at \(a\).