PROBLEM 1 (10 points): Consider the set $X = \{a, b, c\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. 
(a) Is $(X, \mathcal{T})$ a regular space? (b) Is $(X, \mathcal{T})$ a normal space?

PROBLEM 2 (10 points): Prove or disprove: Every path-connected space is locally path-connected.

PROBLEM 3 (10 points): Prove that a completely regular space is regular.

PROBLEM 4 (10 points): Prove or disprove: A second countable topological space is separable.

PROBLEM 5 (10 points): Show that $P^n$ is homeomorphic to the quotient space obtained from the closed unit ball $B^n$ in $\mathbb{R}^n$ by identifying the antipodal points of its boundary $S^{n-1}$.

PROBLEM 6 (10 points): Let $X$ and $Y$ be topological spaces and and let $f : X \to Y$ be an arbitrary map. Prove the following:
$$f : X \to Y \text{ is continuous } \iff \forall A \subseteq X : f(\overline{A}) \subseteq \overline{f(A)}$$

PROBLEM 7 (10 points): If $f : [0, 1] \to [0, 1]$ is a continuous function, then show that $f$ has a fixed point.

PROBLEM 8 (10 points): Show that for any $n > 1$, $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$.

PROBLEM 9 (10 points): Prove or disprove: If there are simply connected open subsets $U$ and $V$ of a topological space $X$ such that $X = U \cup V$ and $U \cap V \neq \emptyset$, then $X$ is simply connected.

PROBLEM 10 (10 points): Let $A$ be a connected subset of a topological space $X$ and let $A \subseteq B \subseteq \overline{A}$.

Show that $B$ is connected.

A subspace $A$ of a topological space $X$ is a retract of $X$ if there is a continuous map $f : X \to A$ such that $f(x) = x$ for all $x \in A$.

PROBLEM 11 (10 points): True or false:
$A = \{z \in \mathbb{C} : |z| = 1\}$ is a retract of $X = \{z \in \mathbb{C} : |z| \leq 1\}$.

PROBLEM 12 (10 points): Let $p : \tilde{X} \to X$ be a covering map, let $x \in X$, and let $\tilde{x}_1, \tilde{x}_2 \in p^{-1}\{x\}$. Then show that $p_*\pi_1(\tilde{X}, \tilde{x}_1)$ and $p_*\pi_1(\tilde{X}, \tilde{x}_2)$ are conjugate in $\pi_1(X, x)$. (Here you may assume that the spaces are Hausdorff, path-connected and locally path-connected.)