MILNOR OPEN BOOKS OF LINKS OF SOME RATIONAL SURFACE SINGULARITIES

MOHAN BHUPAL AND BURAK OZBAGCI

ABSTRACT. We give Legendrian surgery diagrams for canonical contact structures of links of rational surface singularities that are also small Seifert fibred 3-manifolds. For some of these contact structures we construct supporting Milnor open books. Moreover, we describe an infinite family of Milnor fillable contact 3-manifolds so that the Milnor genus (resp. Milnor norm) is strictly greater than the support genus (resp. support norm) of the canonical contact structure, for each member of this family.

1. INTRODUCTION

The link of a normal complex surface singularity carries a canonical contact structure $\xi_{can}$ (a.k.a. the Milnor fillable contact structure) and it is known (cf. [7]) that any Milnor open book on this link supports $\xi_{can}$ in the sense of Giroux [20]. Moreover, $\xi_{can}$ is Stein fillable [5] and hence tight [11]. Furthermore, Lekili and the second author recently showed that $\xi_{can}$ is universally tight [25].

In [17], three numerical invariants of contact structures were studied in terms of open books supporting the contact structures. These invariants are the support genus $sg(\xi)$ (the minimal genus of a page of a supporting open book for $\xi$), the binding number $bn(\xi)$ (the minimal number of binding components of a supporting open book for $\xi$ with minimal genus pages) and the support norm $sn(\xi)$ (minus the maximal Euler characteristic of a page of a supporting open book for $\xi$).

By restricting the domain of open books to only Milnor open books, one can redefine the above invariants specifically for the canonical contact structure $\xi_{can}$ on the link of a complex surface singularity (cf. [1]). We will call these invariants the Milnor genus $Mg(\xi_{can})$, the Milnor binding number $Mb(\xi_{can})$, and the Milnor norm $Mn(\xi_{can})$ of the canonical contact structure $\xi_{can}$. Milnor number, however, is well-known and it corresponds to the first Betti number of the page of the Milnor open book, in our context.

It is clear by definition that $sg(\xi_{can}) \leq Mg(\xi_{can})$ and $sn(\xi_{can}) \leq Mn(\xi_{can})$, but no such inequality exists for the binding numbers in general. In Section 8 we describe an infinite family of Milnor fillable contact 3-manifolds so that $sg(\xi_{can}) < Mg(\xi_{can})$ and $sn(\xi_{can}) < Mn(\xi_{can})$. In fact, we show the existence of a sequence of Milnor fillable
contact 3-manifolds such that \( \text{Mg}(\xi_{\text{can}}) - \text{sg}(\xi_{\text{can}}) \) and \( \text{Mn}(\xi_{\text{can}}) - \text{sn}(\xi_{\text{can}}) \) are arbitrarily large. As a consequence, we deduce that Milnor open books are neither norm nor genus minimizing. We find this result interesting since there are other instances in geometric-topology, where the “complex representatives” are minimizers. Most notably, the link of a complex plane curve singularity bounds a smooth complex curve of genus equal to its Seifert genus.

The examples above are some small Seifert fibred 3-manifolds which are also the links of rational complex surface singularities. In Section 7, we identify the canonical contact structures on all such manifolds, up to isomorphism, via their Legendrian surgery diagrams. Moreover, we describe explicit Milnor open books for canonical contact structures of links of some of these rational surface singularities. Although a Milnor fillable contact structure is supported by infinitely many Milnor open books, it is non-trivial to describe one explicitly. In Section 9, we give many explicit examples of Milnor open books of minimum possible genera.

To fix the notation for the rest of the paper we include a few definitions and facts here. A small Seifert fibred 3-manifold \( Y = Y(e_0; r_1, r_2, r_3) \) is described by its surgery diagram depicted in Figure 1. It is well-known \([32]\) that \( Y \) can also be described by a star-shaped weighted plumbing tree, where the central vertex has weight \( e_0 \) and there are three legs emanating from that vertex corresponding to the continued fraction expansions of the rational numbers \( -\frac{1}{r_i} \), for \( i = 1, 2, 3 \), respectively. Note that, if \( e_0 \leq -3 \), then all tight contact structures on \( Y \) are Stein fillable and they are classified, up to isotopy, in \([39]\).

![Figure 1. Rational surgery diagram for the small Seifert fibred 3-manifold \( Y(e_0; r_1, r_2, r_3) \), where \( e_0 \in \mathbb{Z} \) and \( r_i \in (0, 1) \cap \mathbb{Q} \), for \( i = 1, 2, 3 \).](image)

In the case \( e_0 = -2 \), a classification is available only when \( Y \) is assumed to be an \( L \)-space \([19]\). For the purposes of this paper we will assume that \( Y \) is also the link of some rational complex surface singularity. It follows that \( Y \) is an \( L \)-space by a theorem of Némethi \([29]\). Note that a necessary condition for the 3-manifold \( Y = Y(e_0; r_1, r_2, r_3) \) to be the link of a rational singularity is that \( e_0 \leq -2 \); however, it is not sufficient.

In this paper, we will denote a right-handed (resp. left-handed) Dehn twist along a curve \( \alpha \) as \( \alpha \) (resp. \( \alpha^{-1} \)), for the sake of simplicity. We refer the reader to \([15]\) and \([34]\) for more on open books and contact structures and to \([28]\) for normal surface singularities.
2. OPEN BOOKS AND CONTACT STRUCTURES

Suppose that for an oriented link $B$ in a closed and oriented 3-manifold $Y$, the complement $Y \setminus B$ fibers over the circle as $p: Y \setminus B \to S^1$ such that $p^{-1}(t) = \Sigma_t$ is the interior of a compact surface with $\partial \Sigma_t = B$, for all $t \in S^1$. Then $(B, p)$ is called an open book decomposition (or just an open book) of $Y$. For each $t \in S^1$, the surface $\Sigma_t$ is called a page, while $B$ the binding of the open book. The monodromy of the fibration $p$ is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class of this diffeomorphism is independent of the chosen flow and we will refer to that as the monodromy of the open book decomposition.

Recall that a (positive) contact structure $\xi$ on an oriented 3-manifold is the kernel of a 1-form $\alpha$ such that $\alpha \wedge d\alpha > 0$. In this paper we assume that $\xi$ is coorientable, i.e., $\alpha$ is a global 1–form.

**Definition 1.** An open book decomposition $(B, p)$ of a 3-manifold $Y$ is said to support a contact structure $\xi$ on $Y$ if $\xi$ can be represented by a contact form $\alpha$ such that $\alpha$ evaluates positively on $B$ and $d\alpha$ is a symplectic form on every page.

In [37], Thurston and Winkelnkemper associated a contact structure to every open book. It turns out that the contact structure they constructed is in fact supported by the underlying open book. (Definition 1 was not available at the time.) A converse to the Thurston-Winkelnkemper result is given by

**Theorem 2** (Giroux [20]). Every contact 3-manifold is supported by an open book. Two open books supporting the same contact structure admit a common positive stabilization. Moreover two contact structures supported by the same open book are isotopic.

3. LEGENDRIAN SURGERY DIAGRAMS

Recall that a knot in a contact 3-manifold is called Legendrian if it is everywhere tangent to the contact planes. In order to have a better understanding of the topological constructions in the later sections, we discuss a standard way to visualize Legendrian knots in $S^3$ (actually in $\mathbb{R}^3$) equipped with the standard contact structure $\xi_{st} = \ker(dz + x dy)$. Consider a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st})$ and take its front projection, i.e., its projection to the $yz$-plane. Notice that the projection has no vertical tangencies (since $-\frac{dz}{dy} = x \neq \infty$), and for the same reason at a crossing the strand with smaller slope is in front. It turns out that $L$ can be $C^2$-approximated by a Legendrian knot for which the projection has only transverse double points and cusp singularities (see [18], for example). Conversely, a knot projection with these properties gives rise to a unique Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ — define $x$
from the projection as $x = -\frac{dz}{dy}$. Since any projection can be isotoped to satisfy the above properties, one can easily see that every knot can be isotoped to Legendrian position.

The contact framing $\text{tb}(L)$ of a knot $L$ can be computed as follows. Recall that we measure the contact framing with respect to the Seifert framing in $S^3$. Define $w(L)$ (the \textit{writhe} of $L$) as the sum of signs of the double points — for this to make sense we need to fix an orientation on the knot, but the answer will be independent of this choice. If $c(L)$ is the number of cusps, then the Thurston–Bennequin framing $\text{tb}(L)$ given by the contact structure is equal to $w(L) - \frac{1}{2}c(L)$ with respect to the framing given by a Seifert surface.

Another invariant, the \textit{rotation number} $\text{rot}(L)$ can be defined by trivializing $\xi_{st}$ along $L$ and then taking the winding number of $TL$. For this invariant to make sense we need to orient $L$, and the result will change sign when reversing orientation. Since $H^2(S^3; \mathbb{Z}) = 0$, this number will be independent of the chosen trivialization. For the rotation number we have $\text{rot}(L) = \frac{1}{2}(c_d(L) - c_u(L))$ where $c_d(L)$ (resp. $c_u(L)$) denotes the number of down (resp. up) cusps in the projection.

To describe the Stein fillable contact structures that we deal with in this paper we use Legendrian surgery diagrams as follows: Consider the standard Stein 4-ball $B^4$ with the induced standard contact structure on its boundary. Then attach Weinstein 2-handles (cf. [38]) along an arbitrary Legendrian link in $\partial B^4 = S^3$ to this ball. By the work of Eliashberg [10] the Stein structure on $B^4$ extends over the 2-handles as long as the attaching framing of each 2-handle is $\text{tb} - 1$. The resulting Stein domain has an induced contact structure on its boundary which can be represented by the front projection of the Legendrian link along which we attach the 2-handles. Such a front projection is called a \textit{Legendrian surgery diagram} (see [21] for a thorough discussion). Notice that Legendrian surgery is equivalent to performing contact $(-1)$-surgery along the given Legendrian link in the standard contact $S^3$ (see [9], for example). To describe all Stein fillable contact structures one needs 1-handles as well but those will not appear in our discussion.

4. Milnor open books and the canonical contact structures

Let $(X, x)$ be an isolated normal complex surface singularity. Fix a local embedding of $(X, x)$ in $(\mathbb{C}^N, 0)$. Then a small sphere $S^{2N-1} \subset \mathbb{C}^N$ centered at the origin intersects $X$ transversely, and the complex hyperplane distribution $\xi_{can}$ on $M = X \cap S^{2N-1}$ induced by the complex structure on $X$ is called the \textit{canonical} contact structure. It is known that, for sufficiently small radius $\epsilon$, the contact manifold is independent of $\epsilon$ and the embedding, up to isomorphism. The 3-manifold $M$ is called the link of the singularity and $(M, \xi_{can})$ is called the \textit{contact boundary} of $(X, x)$.

\textbf{Definition 3.} A contact manifold $(Y, \xi)$ is said to be Milnor fillable and the germ $(X, x)$ is called a Milnor filling of $(Y, \xi)$ if $(Y, \xi)$ is isomorphic to the contact boundary $(M, \xi_{can})$ of
some isolated complex surface singularity \((X, x)\).

In addition, we say that a closed and oriented 3-manifold \(Y\) is Milnor fillable if it carries a contact structure \(\xi\) so that \((Y, \xi)\) is Milnor fillable. The contact structure \(\xi\) is called the Milnor fillable contact structure.

By a theorem of Mumford [27], if a contact 3-manifold is Milnor fillable, then it can be obtained by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix. Conversely, it follows from a well-known theorem of Grauert [22] that any 3-manifold that is given by a plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix is Milnor fillable. As for the uniqueness of Milnor fillable contact structures, we have the following fundamental result.

**Theorem 4.** [7] Any closed and oriented 3-manifold has at most one Milnor fillable contact structure up to isomorphism.

In other words, every Milnor fillable 3-manifold \(Y\) has a canonical contact structure \(\xi_{\text{can}}\) which is defined only up to isomorphism. Therefore, in this paper, we identify \(\xi\) and \(\xi_{\text{can}}\), write \((Y, \xi_{\text{can}})\) instead of \((Y, \xi)\) and call \(\xi_{\text{can}}\) as the canonical contact structure.

Since the ground-breaking result of Giroux [20], the geometry of contact structures is often studied via their topological counterparts, namely the open book decompositions. In the realm of surface singularities this fits nicely with some work of Milnor [26].

**Definition 5.** Given an analytic function \(f : (X, x) \rightarrow (\mathbb{C}, 0)\) vanishing at \(x\), with an isolated singularity at \(x\), the open book decomposition \(\text{OB}_f\) of the boundary \(M\) of \((X, x)\) with binding \(L = M \cap f^{-1}(0)\) and projection \(\pi = \frac{f}{|f|} : M \setminus L \rightarrow S^1 \subset \mathbb{C}\) is called the Milnor open book induced by \(f\).

Such functions \(f\) exist and one can talk about many Milnor open books on the singularity link \(M\). Therefore, there are many Milnor open books on any given Milnor fillable contact 3-manifold \((Y, \xi)\), since, by definition, it is isomorphic to the link \((M, \xi_{\text{can}})\) of some isolated complex surface singularity \((X, x)\).

Milnor open books have two essential features [7]: They all support the canonical contact structure \(\xi_{\text{can}}\) and they are horizontal when considered on the plumbing description of a Milnor fillable 3-manifold. Recall that an open book on a plumbing of circle bundles is called *horizontal* if its binding is a collection of some fibres and its pages are transverse to the fibres. One usually requires that the orientation induced on the binding by the pages coincides with the orientation of the fibres induced by the fibration.
Let \((X, x)\) be a germ of a normal complex surface having a singularity at \(x\). Denote by \(X\) a sufficiently small representative of \((X, x)\). Fix a resolution \(\pi: \tilde{X} \to X\) of \((X, x)\) and denote the irreducible components of the exceptional divisor \(E = \pi^{-1}(x)\) by \(\bigcup_{i=1}^{r} E_i\). The fundamental cycle of \(E\) is by definition the componentwise smallest nonzero effective divisor \(Z = \sum z_i E_i\) satisfying \(Z \cdot E_i \leq 0\) for all \(i\).

**Definition 6.** The singularity at \(x\) of the germ \((X, x)\) is called rational if each irreducible component \(E_i\) of the exceptional divisor \(E\) is isomorphic to \(\mathbb{C}P^1\) and

\[
Z \cdot Z + \sum_{i=1}^{n} z_i (-E_i^2 - 2) = -2,
\]

where \(Z = \sum z_i E_i\) is the fundamental cycle of \(E\).

Suppose now that \((X, x)\) is a germ of a normal complex surface having a rational singularity at \(x\). Then we have the following

**Theorem 7.** Both the page-genus and the page-genus plus the number of binding components of the Milnor open book \(\mathcal{O}B_f\) are minimized when \(f\) is taken to be the restriction of a generic linear form on \(\mathbb{C}^N\) to \((X, x)\) for some / any local embedding of \((X, x)\) in \((\mathbb{C}^N, 0)\).

Let \(\mathcal{O}B_{min}\) denote the Milnor open book given by taking the restriction of a generic linear form on \(\mathbb{C}^N\) to \((X, x)\) for some local embedding of \((X, x)\) in \((\mathbb{C}^N, 0)\). We will call \(\mathcal{O}B_{min}\) the minimal Milnor open book. Then it is clear from Theorem 7 that \(Mg(\xi_{can}) = g(\mathcal{O}B_{min})\) and \(Mb(\xi_{can}) = bc(\mathcal{O}B_{min})\), where \(g(\mathcal{O}B)\) (resp. \(bc(\mathcal{O}B)\)) denotes the page-genus (resp. the number of binding components) of the open book \(\mathcal{O}B\). For the Milnor norm, note that, from the definition,

\[
Mn(\xi_{can}) = \min\{2g(\mathcal{O}B) - 2 + bc(\mathcal{O}B)\},
\]

where the minimum is taken over all compatible Milnor open books \(\mathcal{O}B\). Hence it also follows from Theorem 7 that

\[
Mn(\xi_{can}) = 2g(\mathcal{O}B_{min}) - 2 + bc(\mathcal{O}B_{min}) = 2Mg(\xi_{can}) - 2 + Mb(\xi_{can}).
\]

Suppose that \(\pi: \tilde{X} \to X\) is a good resolution of \((X, x)\) and let \(E_1, \ldots, E_r\) denote the irreducible components of the exceptional divisor \(E\). Given an analytic function \(f: (X, x) \to (\mathbb{C}, 0)\) vanishing at \(x\), with an isolated singularity at \(x\), the open book decomposition \(\mathcal{O}B_f\) is a horizontal open book with binding a vertical link of type \(\overline{n} = (n_1, \ldots, n_r)\), where the \(n_i\) are defined as follows: Consider the decomposition \((f \circ \pi) = (f \circ \pi)_e + (f \circ \pi)_s\) of the divisor \((f \circ \pi) \in \text{Div}(\tilde{X})\) into its exceptional and strict parts given by \((f \circ \pi)_e\) is supported
on \( E \) and \( \dim([f\circ\pi]_\ast\cap E) < 1 \). It can be shown that the \( r \)-tuple \( \underline{n} = (n_1, \ldots, n_r) \) satisfies
\[
I(\Gamma)\underline{m}^t = -\underline{n}^t
\]
for some \( r \)-tuple \( \underline{m} = (m_1, \ldots, m_r) \) of positive integers, where \( I(\Gamma) \) denotes the intersection matrix of \( E \).

On the other hand, it follows from Artin \[2\] that for any \( r \)-tuple \( \underline{n} \) of nonnegative integers which satisfies \( 1 \) for some \( r \)-tuple \( \underline{m} \) of positive integers there is a Milnor open book decomposition of the boundary of \( (X, x) \) whose binding is equivalent to a vertical link of type \( \underline{n} \). It can be shown that if \( Z = \sum_{i=1}^{r} z_i E_i \) is the fundamental cycle of the resolution \( \pi \), then the above construction for the \( r \)-tuple \( \underline{m} = (z_1, \ldots, z_r) \) gives the minimal Milnor open book \( \mathcal{OB}_{\text{min}} \).

Remark 8. In \[31\], a generalization of Theorem \[7\] is given for all Milnor fillable rational homology 3-spheres.

6. Planar Milnor Open Books

A vertex in a weighted plumbing graph is called a bad vertex if the sum of the Euler number \( e \) and the degree \( d \) of that vertex is positive.

Proposition 9. Let \( Y \) be the link of a rational surface singularity presented by a weighted plumbing tree \( \Gamma \) with \( r \) vertices. Assume that there are no bad vertices in the plumbing tree.

Then we have \( Mg(\xi_{\text{can}}) = 0, Mb(\xi_{\text{can}}) = -\sum_{i=1}^{r} (e_i + d_i), \) and \( Mn(\xi_{\text{can}}) = -2 + Mb(\xi_{\text{can}}) \).

Proof. The link \( Y \) admits a planar horizontal open book \( \mathcal{OB} \) (cf. \[13\]) with binding a vertical link of type

\[
\underline{n} = -(e_1 + d_1, e_2 + d_2, \ldots, e_r + d_r)
\]

so that for \( \underline{m} = (1, 1, \ldots, 1) \) we have

\[
I(\Gamma)\underline{m}^t = -\underline{n}^t,
\]

where \( I(\Gamma) \) denotes the intersection matrix of the tree \( \Gamma \) which defines \( Y \). Suppose that \( Y \) is the link of the rational surface singularity \( (X, x) \). Then \( \underline{m} \) corresponds to the fundamental cycle of the minimal resolution of \( (X, x) \). Since the open book \( \mathcal{OB} \) and the minimal Milnor open book \( \mathcal{OB}_{\text{min}} \) on the rational homology 3-sphere \( Y \) have equivalent bindings, it follows, by a result of Caubel and Popescu-Pampu \[8\], that \( \mathcal{OB} \) is isotopic to \( \mathcal{OB}_{\text{min}} \).

This proves that \( Mg(\xi_{\text{can}}) = g(\mathcal{OB}) = 0, Mb(\xi_{\text{can}}) = bc(\mathcal{OB}) = -\sum_{i=1}^{r} (e_i + d_i) \) and

\[
Mn(\xi_{\text{can}}) = 2 Mg(\xi_{\text{can}}) - 2 + Mb(\xi_{\text{can}}) = -2 + Mb(\xi_{\text{can}}).
\]

\( \square \)
Remark 10. The canonical contact structure $\xi_{\text{can}}$ of the link of a singularity as in Proposition 9 can be explicitly given by a Legendrian surgery diagram using the methods discussed in [33].

The links of such singularities include lens spaces. Recall that the lens space $L(p, q)$ is obtained from $S^3$ by $-p/q$ surgery on the unknot. Let $[a_1, a_2, \ldots, a_n]$ denote the continued fraction expansion of the rational number $-p/q$, where $a_i \leq -2$, for $i = 1, 2, \ldots, n$. The next result immediately follows from Proposition 9. (Note that a Legendrian surgery diagram for $\xi_{\text{can}}$ on $L(p, q)$ is given by Figure 4 in [33].)

Corollary 11. For the canonical contact structure $\xi_{\text{can}}$ on $L(p, q)$, we have $M_g(\xi_{\text{can}}) = 0$, $M_b(\xi_{\text{can}}) = 2 - 2n - \sum_{i=1}^{n} a_i$, and $M_n(\xi_{\text{can}}) = -2 + M_b(\xi_{\text{can}})$.

In particular, the Milnor binding number and hence the Milnor norm can be made arbitrarily large by choosing, say, $a_1$ arbitrarily small, for fixed $n$. It is known (cf. [36]) that the support genus is zero for all (tight) contact structures on lens spaces. We would like to conjecture that $b_n(\xi_{\text{can}}) = M_b(\xi_{\text{can}})$ and $s_n(\xi_{\text{can}}) = M_n(\xi_{\text{can}})$, for the canonical contact structure $\xi_{\text{can}}$ on $L(p, q)$. This is certainly true for $L(n, n - 1)$, for $n \geq 2$.

The link of singularities in Proposition 9 also include small Seifert fibred manifolds of the form $Y = Y(e_0; r_1, r_2, r_3)$ where $e_0 \leq -3$. Note that in the star-shaped plumbing tree of $Y$ all the weights are less than or equal to $-2$ and therefore one can Legendrian realize these unknots to obtain distinct Stein fillable contact structures on $Y$. The construction in [33], coupled with Wu’s classification [39], allows us to conclude that

Proposition 12. The canonical contact structure $\xi_{\text{can}}$ on $Y = Y(e_0; r_1, r_2, r_3)$, where $e_0 \leq -3$, can be identified as the contact structure obtained by putting all the extra zigzags of the Legendrian unknots in the star-shaped presentation of $Y$, to one fixed side.

7. LEGENDRIAN SURGERY DIAGRAMS FOR CANONICAL CONTACT STRUCTURES

In this section we show that we can relax the assumption “$e_0 \leq -3$” in Proposition 12.

Theorem 13. Let $Y = Y(e_0; r_1, r_2, r_3)$ be a small Seifert fibred 3-manifold which is diffeomorphic to the link of some rational surface singularity. Then the canonical contact structure $\xi_{\text{can}}$ on $Y$ is given, up to isomorphism, by the Legendrian surgery diagram (obtained from the plumbing tree) where one puts all the extra zigzags of the Legendrian unknots to one fixed side.
Proof. Suppose that \((Y, \xi_{can})\) is diffeomorphic to the link of the rational surface singularity \((X, x)\). Then the minimal resolution \(\pi: \tilde{X} \to X\) provides a holomorphic filling \((W, J)\) of \((Y, \xi_{can})\). In particular, \(W\) is a regular neighborhood of the exceptional divisor \(E = \bigcup E_j\) of \(\pi\). Since the curves \(E_j\) are holomorphic, by the adjunction formula we have
\[
\langle c_1(J), [E_j] \rangle = E_j^2 - 2 \text{genus}(E_j) + 2 = E_j^2 + 2.
\]

Now consider the set of tight contact structures \(\xi_i\) on \(Y\). Each \(\xi_i\) is Stein fillable and a Stein filling \((W^i, J^i)\) is given by taking a Legendrian surgery diagram, obtained from the plumbing tree, with the zigzags chosen in a certain way. Denote by \(U_j^i\) the components of the corresponding Legendrian link and by \(S_j^i\) the associated surfaces in the Stein filling \((W^i, J^i)\). Notice that \(W^i\) is diffeomorphic to \(W\) by a diffeomorphism which carries \(S_j^i\) to \(E_j\) for each \(i \) and \(j\). Also, since \(E_j \cdot E_k\) is 0 or 1 if \(j \neq k\), it follows that all components of each of the Legendrian links must have the same orientation.

Now, using the well-known identities
\[
(S_j^i)^2 = \text{tb}(U_j^i) - 1, \quad \langle c_1(J^i), [S_j^i] \rangle = \text{rot}(U_j^i),
\]
observe that \(\langle c_1(J^i), [S_j^i] \rangle = (S_j^i)^2 + 2\) precisely when \(\text{rot}(U_j^i) = \text{tb}(U_j^i) + 1\). Since the latter equality holds exactly when all the cusps of \(U_j^i\) except one are up-cusps, it follows that \(\langle c_1(J^i), [E_j] \rangle = \langle c_1(J^i), [S_j^i] \rangle\) for each \(j\) precisely when all the extra zigzags are chosen so that the additional cusps are all up-cusps, that is, when all the extra zigzags are chosen on the same fixed side (which is determined by the orientation of the Legendrian unknots). The proof is now completed by appealing to the classification of contact structures on small Seifert-fibred 3-manifolds which are diffeomorphic to rational surface singularity links (cf. [19, 39]).

8. Milnor versus support genus

In this section, we describe an infinite family of Milnor fillable contact 3-manifolds so that \(\text{sg}(\xi_{can}) < \text{Mg}(\xi_{can})\) and \(\text{sn}(\xi_{can}) < \text{Mn}(\xi_{can})\). Consider the small Seifert fibred 3-manifold \(Y_p = Y(-2; \frac{1}{3}, \frac{2}{3}, \frac{p}{p+1})\), for \(p \geq 2\). First, we observe that \(Y_p\) is (diffeomorphic to) the link of a rational complex surface singularity, and its resolution graph \(\Gamma_p\) is shown in Figure 2. By the classification of the tight contact structures on \(Y_p\) given by Ghiggini [19], there are exactly two nonisotopic tight contact structures \(\xi_1\) and \(\xi_2\) on \(Y_p\), both of which are Stein fillable.

Proposition 14. For \(i = 1, 2\), we have \(\text{sg}(\xi_i) \leq 1\) and \(\text{sn}(\xi_i) = 2\).

Proof. According to the recipe in [16], first we roll up the plumbing diagram (i.e., we appropriately slide handles) and then Legendrian realize the given surgery curves in a certain way to obtain the Legendrian surgery diagram for a particular Stein fillable contact structure \(\xi_1\) on \(Y_p\) (see Figure 2).
Next, we construct an open book of $Y_p$ supporting this contact structure: We start from an open book of $S^3$ and then embed the surgery curves onto the pages as depicted in Figure 3. The initial page is a torus with one boundary component and the monodromy of this open book of $S^3$ before the surgery is given by $\beta \alpha_1$, where $\alpha_1$ and $\beta$ generate the first homology group of the page. Now we apply the Legendrian surgeries along the given curves to get an open book of $Y_p$ with the monodromy $\phi_p = \alpha_2 \gamma^p \beta \alpha_1 \delta$, where $\delta$ is parallel to the small puncture on the torus, which occurs as a result of stabilizing the page appropriately. Next
we apply some mapping class group tricks. Move $\beta$ over $\gamma^3$ to the left and use the fact that $\gamma\beta = \beta\alpha_1$ to get $\phi_p = \alpha_2\beta\alpha_1^3\beta^p\alpha_1\delta$. Then use the well-known the braid relations and some simple overall conjugations to obtain a more symmetrical presentation of the monodromy

$$\phi_p = (\alpha_2\beta)^2(\alpha_1\beta)^2\beta^{p-2}\delta.$$ 

What we described here is an abstract open book which is compatible with $\xi_1$—where the page is a torus with two boundary components and monodromy is $\phi_p$. Let $\xi_2$ denote the contact structure where we put the extra zigzag in Figure 2 to the left. Note that one can not distinguish the abstract open books corresponding to $\xi$ and $\xi_2$ and in fact $\xi_1$ is isomorphic to $\xi_2$. It follows that $sg(\xi) \leq 1$, since we have already constructed a genus one open book compatible with $\xi$.

One can show that

$$H_1(Y_p, \mathbb{Z}) = \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & p = 2 \mod 3, \\ \mathbb{Z}_9 & \text{otherwise}. \end{cases}$$

By a straightforward calculation, one can show that the Poincaré dual $PD(e(\xi_i)) \in H_1(Y_p, \mathbb{Z})$ of the Euler class $e(\xi_i)$ is a generator of one of the $\mathbb{Z}_3$-factors when $p$ is congruent to 2 mod 3. Similarly $PD(e(\xi_i))$ is a generator of $H_1(Y_p, \mathbb{Z})$ when $p$ is not congruent to 2 mod 3. Therefore the contact structure $\xi_i$ can not be compatible with an elliptic open book with connected binding by Lemma 6.1 in [17], since $e(\xi_i) \neq 0$. Note that $e(\xi_1) = -e(\xi_2)$, which implies that $\xi_1$ is not homotopic to $\xi_2$ as oriented plane fields, although they are isomorphic to each other. In fact, $\xi_2$ is obtained by $\xi_1$ by reversing the orientation of the underlying plane field.

Now we claim that $sn(\xi_i) = 2$. To prove our claim we need to exclude the possibility that $\xi_i$ is compatible with a planar open book with less than four binding components. Suppose that $\xi_i$ is compatible with a planar open book, i.e., $sg(\xi_i) = 0$. If $bn(\xi_i) \leq 2$, then $\xi_i$ is the unique tight contact structure on the lens space $L(n, n-1)$ for some $n \geq 0$ (cf. [17]) which is indeed impossible since $Y_p$ is not a lens space.

Next we rule out the possibility that $bn(\xi_i) = 3$. Let $\Sigma$ be the planar surface with three boundary components. Any diffeomorphism of $\Sigma$ is determined by three numbers $q, r, s$, that give the number of Dehn twists on curves $\tau_1, \tau_2, \tau_3$ parallel to each boundary component. Let $Y_{q,r,s}$ be the 3-manifold determined by the open book with page $\Sigma$ and monodromy given by $\tau_1^q\tau_2^r\tau_3^s$. It is easy to see that $Y_{q,r,s}$ is the Seifert fibred space shown in Figure 4.

One can compute that $|H_1(Y_{q,r,s}, \mathbb{Z})| = qr + qs + rs$. Suppose that $\xi_i$ is compatible with an open book with page $\Sigma$ and monodromy $\tau_1^q\tau_2^r\tau_3^s$. The tightness of $\xi_i$ implies that the integers $q, r, s$ are all nonnegative, because otherwise $\tau_1^q\tau_2^r\tau_3^s$ is not right-veering [23]. Moreover, since the order of the first homology group of $Y_p$ is 9, for all $p \geq 2$, we conclude
that \((g, r, s)\) is equal to either \((0, 1, 9)\), \((0, 3, 3)\) or \((1, 1, 4)\). Hence \(Y_p\) is diffeomorphic to either \(L(9, 8)\), \(L(3, 2)\#L(3, 2)\) or \(L(9, 4)\), which is a contradiction. Hence, \(b_n(\xi_i) \geq 4\). This finishes the proof of our claim that \(s_n(\xi_i) = 2\).

One can ask whether or not \(s_g(\xi_i) = 1\), although it is not essential for the purposes of this paper. There are two known methods (cf. [14, 35]) of finding obstructions to planarity of a contact structure, but unfortunately both fail in our case. That is because \(Y_p\) is an \(L\)-space and it is not an integral homology sphere.

**Proposition 15.** We have \(\text{Mg}(\xi_{can}) = 2\) and \(\text{Mn}(\xi_{can}) = 3\).

**Proof.** Enumerate the vertices of the plumbing graph from left to right along the top row with the bottom vertex coming last. It is then easy to check that the \((p + 4)\)-tuple of positive integers \(m\) corresponding to the fundamental cycle of the minimal resolution of the singularity of which \(Y_p\) is the link is given by \(m = (1, 2, 3, 3, \ldots, 3, 3, 2, 1, 1)\). The construction in [4] now gives an open book decomposition \(\text{OB}(m) = \text{OB}_{min}\) of \(Y_p\) with binding a vertical link of type \(n = (0, 0, 1, 0, \ldots, 0)\). Note that here \(m\) and \(n\) are related by

\[I(\Gamma_p)m^t = -n^t.\]

Using the formula

\[g(\text{OB}(m)) = 1 + \sum_{i=1}^{p+4} \frac{(v_i - 2)m_i + (m_i - 1)n_i}{2}\]

given in Lemma 3.1 in [1], where \(m = (m_1, \ldots, m_{p+4})\) and \(n = (n_1, \ldots, n_{p+4})\), one has \(\text{Mg}(\xi_{can}) = g(\text{OB}(m)) = 2\) for the unique Milnor fillable contact structure \(\xi_{can}\) on each 3-manifold \(Y_p\). Also one has \(\text{Mb}(\xi_{can}) = bc(\text{OB}(m)) = \sum_{i=1}^{p+4} n_i = 1\). It follows that \(\text{Mn}(\xi_{can}) = 3\), completing the proof of the proposition.

\[\square\]
\[ -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \]
\[
\begin{array}{c}
\text{n vertices} \\
\bigcirc \\
\text{n vertices}
\end{array}
\]
\[ -(n + 1) \]

**Figure 5.** The plumbing graph for \( P_n \)

Now since any Milnor fillable contact structure is Stein fillable \([5]\), \( \xi_{\text{can}} \) has to be isomorphic to \( \xi_i \) by Ghiggini’s classification \([19]\). Note that it does not make sense to distinguish \( \xi_1 \) and \( \xi_2 \) here since they are isomorphic to each other. Thus

**Corollary 16.** We have \( \text{sg}(\xi_{\text{can}}) < \text{Mg}(\xi_{\text{can}}) \) and \( \text{sn}(\xi_{\text{can}}) < \text{Mn}(\xi_{\text{can}}) \).

Note, however, that \( \text{Mb}(\xi_{\text{can}}) = 1 \) while \( \text{bn}(\xi_{\text{can}}) \geq 2 \).

The arguments above can be generalized to prove the following

**Theorem 17.** For each positive integer \( k \), there is a Milnor fillable contact 3-manifold such that \( \text{Mg}(\xi_{\text{can}}) - \text{sg}(\xi_{\text{can}}) \geq k \) and \( \text{Mn}(\xi_{\text{can}}) - \text{sn}(\xi_{\text{can}}) \geq k \).

**Proof.** Consider the small Seifert fibred 3-manifolds \( P_n = Y(-2, \frac{1}{y+1}, \frac{n}{n+1}, \frac{n}{n+1}) \), for \( n \geq 2 \). Each of these is (diffeomorphic to) the link of a rational complex surface singularity with minimal resolution graph the weighted tree shown in Figure 5. By the classification of tight contact structures on \( P_n \) \([19]\) there are exactly \( n \) nonisotopic tight contact structures \( \xi_i \), for \( i = 1, \ldots, n \), on \( P_n \), each of which is Stein fillable. By \([16]\), for each of these tight contact structures \( \xi_i \) we can find a genus one supporting open book. Since the canonical contact structure \( \xi_{\text{can}} \) on \( P_n \) is tight it must be isomorphic to one of the contact structures \( \xi_i \). This proves that \( \text{sg}(\xi_{\text{can}}) \leq 1 \). It is easy to see that \( \text{sn}(\xi_{\text{can}}) \leq n \), since each \( \xi_i \) is supported by an elliptic open book with \( n \) binding components.

Now enumerate the vertices of the graph in Figure 5 as before and consider the \((2n + 2)\)-tuple \( m = (1, 2, 3, \ldots, n - 1, n, n + 1, n, n - 1, \ldots, 3, 2, 1, 1) \) of positive integers. This corresponds to the fundamental cycle of the minimal resolution of the singularity of which \( P_n \) is the link. It now follows that \( \text{Mg}(\xi_{\text{can}}) = n \), \( \text{Mn}(\xi_{\text{can}}) = 2n - 1 \), \( \text{sg}(\xi_{\text{can}}) \leq 1 \), and \( \text{sn}(\xi_{\text{can}}) \leq n \), proving the theorem, by taking \( k = n - 1 \). \( \square \)

An immediate consequence is that

**Corollary 18.** Not every open book on the link of a surface singularity is isomorphic to a Milnor open book.
In this section, we consider a family of rational surface singularities whose links are diffeomorphic to some small Seifert fibred 3-manifold of the form \( M_k = Y(-2; \frac{4}{3}, \frac{1}{3}, \frac{2k+1}{2k+3}) \), for \( k \geq 0 \) as depicted in Figure 6. Note that this resolution graph comes from a rational surface singularity.

First we roll up the plumbing diagram and then Legendrian realize the given surgery curves in a certain way to obtain a particular Stein fillable contact structure on \( M_k \). Next, we construct an open book of \( M_k \) supporting this contact structure by the method discussed in [16]: We start from an open book of \( S^3 \), where the page is a torus with one boundary component and the monodromy is given by \( \beta \alpha_2 \). Then embed the surgery curves onto the pages as depicted in Figure 7.

**Figure 6.** The canonical contact structure \( \xi_{\text{can}} \) on \( M_k \), for \( k \geq 0 \)
The page of the open book on $M_k$ is a torus with four punctures (see Figure 7) and the monodromy of the open book after the surgery is given by

$$\phi_k = \alpha_1 \gamma_2 \gamma_1^{k+1} \beta_1 \beta \alpha_2 \delta_1 \delta_2 \delta_3$$

$$= \alpha_1 \beta \alpha_4 \alpha_3^{k+1} \beta_1 \alpha_2 \alpha_2^{-1} \alpha_3 \delta_1 \delta_2 \delta_3$$

$$= \alpha_1 \beta \alpha_4 \alpha_3^{k+1} \beta_1 \alpha_2 \alpha_2^{-1} \beta \alpha_3 \delta_1 \delta_2 \delta_3$$

$$= \alpha_1 \beta \alpha_4 \alpha_3^{k+1} \alpha_2 \alpha_2^{-1} \beta \alpha_3 \delta_1 \delta_2 \delta_3$$

$$= \alpha_1 \beta \alpha_4 \alpha_3^{k+1} \alpha_2 \alpha_2^{-1} \alpha_3 \delta_1 \delta_2 \delta_3$$

$$= \alpha_1 \beta \alpha_4 \alpha_3^{k+1} \alpha_2 \alpha_2^{-1} \beta \alpha_3 \delta_1 \delta_2 \delta_3.$$

Note that we moved $\beta$ over $\gamma_1$ and $\gamma_2$ to the left in the second equation above, naming the resulting curves $\alpha_3$ and $\alpha_4$, respectively. Also we moved $\alpha_2 \alpha_3^{-1}$ to the left over $\beta_1$ in the fourth equation. By using braid relations and simple induction we obtain that

$$\phi_k = \alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \alpha_3^k \beta \delta_1 \delta_2 \delta_3$$

for any $0 \leq s \leq k$.

Rename $\alpha_i$ as $\alpha_{i+1}$ mod 4, to cyclically permute the $\alpha$ curves on the torus. It follows that

$$\phi_k = \begin{cases} 
\delta_1 \delta_2 \delta_3 \alpha_2^{(k-1)/2} \alpha_4^{(k-1)/2} \alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \beta_2 & \text{when } k \text{ is odd and } s = (k+1)/2, \\
\delta_1 \delta_2 \delta_3 \alpha_2^{k/2} \alpha_4^{k/2} \alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \beta & \text{when } k \text{ is even and } s = k/2.
\end{cases}$$

Now we claim that this particular open book on $M_k$ agrees with a Milnor open book. To see this, enumerate the vertices of the plumbing graph from left to right along the top row with the bottom vertex coming last and consider the $(k+4)$-tuple of positive integers $m = (1,2,\ldots,2,1,1)$. The page of the associated Milnor open book, which is $OB_{min}$, is depicted in Figure 8.

From the construction in [4], the monodromy $\psi_k$ satisfies

$$\psi_k^2 = \delta_1^3 \delta_2^3 \delta_3^4 \alpha_2^k \alpha_4^k.$$
We claim $\psi_k = \phi_k$. To see this, using the uniqueness result from [6], clearly it is sufficient to check that

$$ (\delta_1 \delta_2 \delta_3 \alpha_3 \beta \alpha_4 \delta_4 \alpha_4^{k-s})^2 = \delta_1^3 \delta_2^3 \delta_3 \delta_4 \alpha_2^k \alpha_4^k $$

for any $0 \leq s \leq k$. Cancelling $\delta_2^3 \delta_3 \delta_4$ on both sides and using the four-holed torus relation $(\alpha_1 \alpha_3 \beta) (\alpha_2 \alpha_4 \beta) = \delta_1^3 \delta_2^3 \delta_3 \delta_4 \alpha_4^k \alpha_4^k$ (cf. [24]), we thus need to check

$$ \alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \delta_4 \alpha_4^{k-s} \alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \beta \alpha_4 \delta_4 \alpha_4^{k-s} = (\alpha_1 \alpha_3 \beta \alpha_2 \alpha_4 \beta)^2 \alpha_2^k \alpha_4^k. $$

Making further cancellations, it is thus sufficient to check that

$$ \alpha_4^{k-s} \beta \alpha_2 \alpha_4 \alpha_4 \delta_4 \alpha_4^{k-s} = \beta \alpha_2 \alpha_4 \alpha_4 \delta_4 \alpha_4^k $$

for any $0 \leq s \leq k$, which can be done by using the braid relations and a simple induction argument.

Note that there are exactly eight (cf. [19]) distinct tight contact structures on $M_k$. The contact structure in Figure 6 is isomorphic to $\xi_{\text{can}}$ on $M_k$. Putting all the extra zigzags to the left rather than to the right in Figure 6 would yield an isomorphic contact structure.

10. Final Remarks

The minimal Milnor open book $\mathcal{OB}_{\text{min}}$ on $Y = Y(e_0; r_1, r_2, r_3)$ realizes $\text{Mg}(\xi_{\text{can}})$, $\text{Mb}(\xi_{\text{can}})$ and $\text{Mn}(\xi_{\text{can}})$. In fact, it follows from the proof of Theorem 7 given in [11] that $\mathcal{OB}_{\text{min}}$ is the unique Milnor open book that realizes $\text{Mg}(\xi_{\text{can}})$, $\text{Mb}(\xi_{\text{can}})$ and $\text{Mn}(\xi_{\text{can}})$. Thus any other Milnor open book on $Y$ that realizes $\text{Mg}(\xi_{\text{can}})$ cannot realize $\text{Mb}(\xi_{\text{can}})$ and $\text{Mn}(\xi_{\text{can}})$. For example, consider the 3-manifold $Y = Y(-2; 1/2, 1/2)$, which is the link of the singularity $D_4$. The pages of two Milnor open books on $Y$ are given in Figure 9. The first one is the minimal Milnor open book $\mathcal{OB}_{\text{min}} = \mathcal{OB}((1, 2, 1, 1))$ with page a once-punctured torus and monodromy $\phi = (\alpha \beta)^3$; the second one is the Milnor open book $\mathcal{OB} = \mathcal{OB}((2, 2, 1, 1))$ with page a twice-punctured torus and monodromy $\psi$ satisfying $\psi^2 = \delta_1 \delta_2 \alpha_2^2 \beta \alpha_2$. Using the uniqueness result from [6] and the two-holed torus relation one can check that $\psi = \alpha_1 \alpha_2 \beta \alpha_2^2 \beta \alpha_2$. It is easy to see that $\mathcal{OB}$ is related to $\mathcal{OB}_{\text{min}}$ by a single positive stabilization.
There are Milnor fillable contact 3-manifolds such that
\[ \text{sg}(\xi_{can}) = \text{Mg}(\xi_{can}), \text{sn}(\xi_{can}) = \text{Mn}(\xi_{can}), \text{and } \text{bn}(\xi_{can}) = \text{Mb}(\xi_{can}). \]

For instance, the unique tight contact structure on the link of the singularity \( E_8 \) (with negative definite intersection form) satisfies all the equalities above. The invariants \( \text{sg}(\xi), \text{sn}(\xi) \) and \( \text{bn}(\xi) \) are independent in general for a contact structure \( \xi \) as illustrated in [3] (and [12]) although
\[ \text{Mn}(\xi_{can}) = 2 \text{Mg}(\xi_{can}) - 2 + \text{Mb}(\xi_{can}). \]

There are examples of canonical contact structures such that Milnor genus, Milnor norm and Milnor binding number are arbitrarily large as we showed in Proposition 9 and Theorem 17.

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**REFERENCES**


Department of Mathematics, METU, Ankara, Turkey

E-mail address: bhupal@metu.edu.tr

Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA, 94720-5070

E-mail address: bozbagci@msri.org