On the relative Giroux correspondence

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Abstract. Recently, Honda, Kazez and Matić described an adapted partial open book decomposition of a compact contact 3-manifold with convex boundary by generalizing the work of Giroux in the closed case. They also implicitly established a one-to-one correspondence between isomorphism classes of partial open book decompositions modulo positive stabilization and isomorphism classes of compact contact 3-manifolds with convex boundary. In this expository article we explicate the relative version of Giroux correspondence.

1. Introduction

Let \((M, \Gamma)\) be a balanced sutured 3-manifold and let \(\xi\) be a contact structure on \(M\) with convex boundary whose dividing set on \(\partial M\) is isotopic to \(\Gamma\). Recently, Honda, Kazez and Matić [HKM09] introduced an invariant of the contact structure \(\xi\) which lives in the sutured Floer homology group defined by Juhász [Ju]. This invariant is a relative version of the contact class in Heegaard Floer homology in the closed case as defined by Ozsváth and Szabó [OzSz] and reformulated in [HKM07]. Both the original definition in [OzSz] and the reformulation of the contact class by Honda, Kazez and Matić are based on the so-called Giroux correspondence [Gi02] which is a one-to-one correspondence between open book decompositions modulo positive stabilization and isotopy classes of contact structures on closed 3-manifolds.

In order to adapt their reformulation [HKM07] of the contact class to the case of a contact manifold \((M, \xi)\) with convex boundary, Honda, Kazez and Matić described in [HKM09], a partial open book decomposition of \(M\) (adapted to \(\xi\)) by generalizing the work of Giroux in the closed case. This description coupled with Theorem 1.2 (and the subsequent discussion) in [HKM09] induces a map from isomorphism classes of contact 3-manifolds with convex boundary to isomorphism classes of partial open book decompositions modulo positive stabilization. Here we spell out the inverse of this map, by describing a compact contact manifold with convex boundary, compatible contact structure.
3-manifold with convex boundary compatible with an abstract partial open book decomposition. To define a contact structure compatible with an abstract partial open book decomposition we chose to mimic the analogous result of Torisu [To] (rather than adapting the construction of Thurston and Winkelnkemper [ThWi]) which conveniently allowed us to keep track of the dividing set on the boundary. Consequently, one obtains a relative version of Giroux correspondence which is due to Honda, Kazez and Matić.

**Theorem 1.1.** There is a one-to-one correspondence between isomorphism classes of partial open book decompositions modulo positive stabilization and isomorphism classes of compact contact 3-manifolds with convex boundary.

The relative Giroux correspondence helps understand the geometric properties of contact 3-manifolds using partial open books, e.g. if the monodromy of a corresponding partial open book is not right-veering, then the contact structure is overtwisted. It also plays a critical role in the definition of the (relative) contact invariant in sutured Floer homology which helps to analyze the contact invariant of a closed manifolds in terms of the relative contact invariants of certain compact pieces. In [GHV], it is proved that the contact invariant vanishes in the presence of Giroux torsion using some properties of the relative invariant.

The paper is organized as follows: In Section 2 we give the definition of an abstract partial open book decomposition \((S, P, h)\), construct a balanced sutured manifold \((M, \Gamma)\) associated to \((S, P, h)\), and construct a (unique) compatible contact structure \(\xi\) on \(M\) which makes \(\partial M\) convex with a dividing set isotopic to \(\Gamma\). In Section 3 we prove Theorem 1.1 after reviewing the related results due to Honda, Kazez and Matić [HKM09]. In the last section we provide examples of abstract partial open books compatible with some basic contact 3-manifolds with boundary.

The reader is advised to turn to Etnyre’s notes [Etn] for the related material on contact topology of 3-manifolds.

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2. Partial open books, sutured manifolds and contact structures

**Definition 2.1.** An abstract partial open book decomposition is a triple \((S, P, h)\) satisfying the following conditions:

1. \(S\) is a compact oriented connected surface with \(\partial S \neq \emptyset\),
2. \(P = P_1 \cup P_2 \cup \ldots \cup P_r\) is a proper (not necessarily connected) subsurface of \(S\) such that \(S\) is obtained from \(S \setminus P\) by successively attaching 1-handles \(P_1, P_2, \ldots, P_r\),
3. \(h : P \to S\) is an embedding such that \(h|_A = \text{identity}\), where \(A = \partial P \cap \partial S\).

**Remark.** Figures 1 and 2 present simple examples of partial open book decompositions. It follows from the definition that \(A\) is a 1-manifold with nonempty boundary (but it may have closed components as in Figure 4) and \(\partial P \setminus A\) is a nonempty set consisting of some arcs (but no closed components). The connectedness condition on \(S\) is not essential, but simplifies the discussion.
We now briefly turn our attention to sutured manifolds which was introduced by Gabai [Ga] to study foliations. A sutured manifold \((M, \Gamma)\) is a compact oriented 3-manifold with nonempty boundary, together with a compact subsurface \(\Gamma = A(\Gamma) \cup T(\Gamma) \subset \partial M\), where \(A(\Gamma)\) is a union of pairwise disjoint annuli and \(T(\Gamma)\) is a union of tori. Moreover each component of \(\partial M \setminus \Gamma\) is oriented, subject to the condition that whether or not the orientation agrees with the orientation induced as the boundary of \(M\) changes every time we nontrivially cross \(A(\Gamma)\). Let \(R_+(\Gamma)\) (resp. \(R_-(\Gamma)\)) be the open subsurface of \(\partial M \setminus \Gamma\) on which the orientation agrees with (resp. is the opposite of) the boundary orientation on \(\partial M\). A sutured manifold \((M, \Gamma)\) is balanced if \(M\) has no closed components, \(\pi_0(A(\Gamma)) \to \pi_0(\partial M)\) is surjective, and \(\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))\) on every component of \(M\). It turns out that if \((M, \Gamma)\) is balanced, then \(\Gamma = A(\Gamma)\) and every component of \(\partial M\) nontrivially intersects \(\Gamma\). Since all the sutured manifolds that we will deal with in this paper are balanced, we will think of \(\Gamma\) as a set of oriented curves on \(\partial M\) by identifying each annulus in \(\Gamma\) with its core circle. Here we orient \(\Gamma\) as the boundary of \(R_+(\Gamma)\).

We now emphasize the relation between dividing sets and sutures. Let \(\xi\) be a contact structure on a compact oriented 3-manifold \(M\) whose dividing set on the convex boundary \(\partial M\) is denoted by \(\Gamma\). Then it is fairly easy to see that \((M, \Gamma)\) is a balanced sutured manifold (with annular sutures) via the identification we mentioned above. Conversely, given a balanced sutured manifold \((M, \Gamma)\), there exists a contact structure \(\xi\) on \(M\) which makes \(\partial M\) convex and realizes \(\Gamma\) as its diving set on \(\partial M\). However one should keep in mind that the contact structure is not uniquely determined and cannot always be chosen to be tight.
Given a partial open book decomposition \((S, P, h)\), we construct a sutured manifold \((M, \Gamma)\) as follows: Let

\[ H = (S \times [-1, 0]) / \sim \]

where \((x, t) \sim (x, t')\) for \(x \in \partial S\) and \(t, t' \in [-1, 0]\). It is easy to see that \(H\) is a solid handlebody whose oriented boundary is the surface \(S \times \{0\} \cup -S \times \{-1\}\) (modulo the relation \((x, 0) \sim (x, -1)\) for every \(x \in \partial S\)). Similarly let

\[ N = (P \times [0, 1]) / \sim \]

where \((x, t) \sim (x, t')\) for \(x \in A\) and \(t, t' \in [0, 1]\). Since \(P\) is not necessarily connected \(N\) is not necessarily connected. Observe that each component of \(N\) is also a solid handlebody. The oriented boundary of \(N\) can be described as follows: Let the arcs \(c_1, c_2, \ldots, c_n\) denote the connected components of \(\partial P \setminus A\). Then, for \(1 \leq i \leq n\), the disk \(D_i = (c_i \times [0, 1]) / \sim\) belongs to \(\partial N\). Thus part of \(\partial N\) is given by the disjoint union of \(D_i\)'s. The rest of \(\partial N\) is the surface \(P \times \{1\} \cup -P \times \{0\}\) (modulo the relation \((x, 0) \sim (x, 1)\) for every \(x \in A\)).

Let \(M = N \cup H\) where we glue these manifolds by identifying \(P \times \{0\} \subset \partial N\) with \(P \times \{0\} \subset \partial H\) and \(P \times \{1\} \subset \partial N\) with \(h(P) \times \{-1\} \subset \partial H\). Since the gluing identification is orientation reversing \(M\) is a compact oriented 3-manifold with oriented boundary

\[ \partial M = (S \setminus P) \times \{0\} \cup -((S \setminus h(P)) \times \{-1\} \cup (\partial P \setminus A) \times [0, 1]) \]

(modulo the identifications given above).

**Definition 2.2.** If a compact 3-manifold \(M\) with boundary is obtained from \((S, P, h)\) as discussed above, then we call the triple \((S, P, h)\) a partial open book decomposition of \(M\).

We define the suture \(\Gamma\) on \(\partial M\) as the set of closed curves (see Remark 2) obtained by gluing the arcs \(c_i \times \{1/2\} \subset \partial N\), for \(1 \leq i \leq n\), with the arcs in

![Figure 3. A partial open book decomposition: M as the union of N and H](image-url)
\((\partial S \setminus \partial P) \times \{0\} \subset \partial H\), hence as an oriented simple closed curve and modulo identifications
\[
\Gamma = (\partial S \setminus \partial P) \times \{0\} \cup -\partial P \setminus \Lambda \times \{1/2\}.
\]

**Remark.** If a sutured manifold \((M, \Gamma)\) has only annular sutures, then it is convenient to refer to the set of core circles of these annuli as \(\Gamma\).

**Definition 2.3.** The sutured manifold \((M, \Gamma)\) obtained from a partial open book decomposition \((S, P, h)\) as described above is called the sutured manifold associated to \((S, P, h)\).

**Definition 2.4 ([Ju]).** A sutured manifold \((M, \Gamma)\) is balanced if \(M\) has no closed components, \(\pi_0(A(\Gamma)) \to \pi_0(\partial M)\) is surjective, and \(\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))\) on every component of \(\partial M\).

**Remark.** It follows that if \((M, \Gamma)\) is balanced, then \(\Gamma = A(\Gamma)\) and every component of \(\partial M\) nontrivially intersects the suture \(\Gamma\).

**Lemma 2.5.** The sutured manifold \((M, \Gamma)\) associated to a partial open book decomposition \((S, P, h)\) is balanced.

**Proof.** It is clear that \(M\) is connected since we assumed that \(S\) is connected. We observe that \(\partial M \neq \emptyset\) since \(P\) is a proper subset of \(S\) by our definition. In fact, \(\partial M\) can be described starting from the connected surface \(\partial H = S \times \{0\} \cup S \times \{-1\}\): Let \(\kappa_j\) be \(a_j \cup b(a_j)\), where \(a_j\) is the cocore of the 1-handle \(P_i\) in \(P\) (see Figure 4 for suitable \(a_j\)'s). Then \(\partial M\) is obtained by cutting \(\partial H\) along \(\kappa_j\)'s and capping off each resulting boundary by a disk \(D_i = (c_i \times [0,1])/\sim\) for some \(i\). From this description it is clear that every component of \(\partial M\) contains a \(c_i \times \{1/2\} \subset M\) and therefore \(\pi_0(A(\Gamma)) \to \pi_0(\partial M)\) is surjective. Now let \(R_+(\Gamma)\) be the open subsurface in \(\partial M\) obtained by gluing
\[
((S \setminus \partial S) \setminus P) \times \{0\} \subset \partial H \quad \text{and} \quad \bigcup_{i=1}^n (c_i \times [0,1/2])/\sim \subset \partial N
\]
and \(R_-(\Gamma)\) be the open subsurface in \(\partial M\) obtained by gluing
\[
((S \setminus \partial S) \setminus h(P)) \times \{-1\} \subset \partial H \quad \text{and} \quad \bigcup_{i=1}^n (c_i \times (1/2,1])/\sim \subset \partial N
\]
under the gluing map that is used to construct \(M\). Since \(h : P \to S\) is an embedding we have \(\chi(P) = \chi(h(P))\) and it follows that \(\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))\).

The following result is inspired by Torisu's work [To] in the closed case.

**Proposition 2.6.** Let \((M, \Gamma)\) be the balanced sutured manifold associated to a partial open book decomposition \((S, P, h)\). Then there exists a contact structure \(\xi\) on \(M\) satisfying the following conditions:

1. \(\xi\) is tight when restricted to \(H\) and \(N\).
2. \(\partial H\) is a convex surface in \((M, \xi)\) whose dividing set is \(\partial S \times \{0\}\).
3. \(\partial N\) is a convex surface in \((M, \xi)\) whose dividing set is \(\partial P \times \{1/2\}\).

Moreover such \(\xi\) is unique up to isotopy.

**Proof.** We will prove that there is a unique tight contact structure (up to isotopy) on \(H\) and \(N\) with the given boundary conditions, using arguments along similar lines.\(^1\) Once we have these contact structures on \(H\) and \(N\), since the dividing

\[^1\]In fact, one can prove a general existence and uniqueness theorem using an explicit contact form \(\lambda + dt\) on \(\Sigma \times [0,1]/\sim\), for any surface \(\Sigma\) with boundary, where \(\lambda\) is a primitive of a volume form on \(\Sigma\) that is standard near the boundary. It can be argued that this contact form gives a tight contact structure making the boundary convex with dividing set \(\partial \Sigma \times \{1/2\}\).
sets on $\partial H$ and $\partial N$ agree on the subsurface along which we glue $H$ and $N$, we obtain a unique contact structure (up to isotopy) on $M$ satisfying the above conditions, by gluing together the contact structures on these pieces.

To prove the existence of tight contact structures on $H$ and $N$ with prescribed dividing sets we simply consider $H$ and $N$ embedded in the closed contact $3$-manifold $(Y, \xi')$ supported by the open book $(S, id)$ and appeal to the closed case (see [To] and [Etn, Lemma 4.4]). For $H$, observe that

$$H = (S \times [-1, 0])/\sim \subset (S \times [-1, 1])/\sim = Y,$$

where the equivalence relation $\sim$ is given by, $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 1]$, and $(s, -1) \sim (s, 1)$ for $s \in S$. The contact structure $\xi'$ is Stein fillable by [Gi02], hence tight by [ElGr], and therefore its restriction to $H$ is also tight. In fact, $\partial H$ is convex with respect to $\xi'$ with dividing set $\partial S \times \{0\}$ (see Lemma 4.4 in [Etn]). Similarly, $N$ trivially embeds in $H$ since $\partial P \times \{1/2\}$ is the union of $A \times \{0\}$ and the arcs $c_i \times \{1/2\}$, for $1 \leq i \leq n$. So $\xi'$ restricts to a tight contact structure on $N$. To identify its dividing set we first observe that the dividing set on $P \times \{1\} \cup -P \times \{0\} = \partial N \cap \partial H$ is the set $A \times \{0\} = \partial N \cap (\partial S \times \{0\})$. The rest of $\partial N$ consists of the disks $D_i = (c_i \times [0, 1]) / \sim$. Each one of these disks can be made convex so that the dividing set is a single arc since its boundary intersects the dividing set twice. It follows that the dividing set on $\partial N$ is as required after rounding the edges.

In order to prove the uniqueness for $H$, as in Lemma 4.4 in [Etn], we take a set \{$d_1, \ldots, d_p$\} of properly embedded pairwise disjoint arcs in $S$ whose complement is a single disk. (It follows that the set \{$d_1, d_2, \ldots, d_p$\} represents a basis of $H_1(S, \partial S)$.) For $1 \leq k \leq p$, let $\delta_k$ denote the closed curve on $\partial H$ which is obtained by gluing the arc $d_k$ on $S \times \{0\}$ with the arc $d_k$ on $S \times \{-1\}$. Then we observe that \{$\delta_1, \delta_2, \ldots, \delta_p$\} is a set of homologically linearly independent closed curves on $\partial H$ so that $\delta_k$ bounds a compressing disk $D_k = (d_k \times [0, -1]) / \sim$ in $H$. It is clear that when we cut $H$ along $D_k$’s (and smooth the corners) we get a 3-ball $B^3$. Moreover $\delta_k$ intersects the dividing set twice by our construction. Now we put each $\delta_k$ into Legendrian position (by the Legendrian realization principle [Ho0]) and make the compressing disk $D_k$ convex [G191]. The dividing set on $D_k$ will be an arc connecting two points on $\partial D_k = \delta_k$. Then we cut along these disks and round the edges (see [H00]) to get a connected dividing set on the remaining $B^3$. Consequently, Theorem 2.7 due to Eliashberg (although stated in different terms in [El]) implies the uniqueness of a tight contact structure on $H$ with the assumed boundary conditions. Recall that a standard contact 3-ball is a tight contact 3-ball with convex boundary whose dividing set is connected.

**Theorem 2.7** (Eliashberg). There is a unique standard contact 3-ball.

The proof of the uniqueness of such a tight contact structure on $N$ follows a similar line. Instead of a basis of $H_1(S, \partial S)$ we take suitable cocores \{$a_1, \ldots, a_r$\} of the 1-handles $P_j$’s in $P$ to get a basis of $H_1(P, A)$ (see Figure 4 for an example). Then one can proceed as in the proof given above for the handlebody $H$. \qed

Proposition 2.6 leads to the following definition of compatibility of a contact structure and a partial open book decomposition.

**Definition 2.8.** Let $(M, \Gamma)$ be the balanced sutured manifold associated to a partial open book decomposition $(S, P, h)$. A contact structure $\xi$ on $(M, \Gamma)$ is
said to be compatible with \((S, P, h)\) if it is isotopic to a contact structure satisfying conditions (1), (2) and (3) stated in Proposition 2.6.

**Definition 2.9.** Two partial open book decompositions \((S, P, h)\) and \((\tilde{S}, \tilde{P}, \tilde{h})\) are isomorphic if there is a diffeomorphism \(f : S \to \tilde{S}\) such that \(f(P) = \tilde{P}\) and \(\tilde{h} = f \circ h \circ (f^{-1})|_{\tilde{P}}\).

**Remark.** It follows from Proposition 2.6 that every partial open book decomposition has a unique compatible contact structure, up to isotopy, on the balanced suture manifold associated to it, such that the dividing set of the convex boundary is isotopic to the suture. Moreover if \((S, P, h)\) and \((\tilde{S}, \tilde{P}, \tilde{h})\) are isomorphic partial open book decompositions, then the associated compatible contact 3-manifolds \((M, \Gamma, \xi)\) and \((\tilde{M}, \tilde{\Gamma}, \tilde{\xi})\) are also isomorphic.

**Definition 2.10.** Let \((S, P, h)\) be a partial open book decomposition. A partial open book decomposition \((S', P', h')\) is called a positive stabilization of \((S, P, h)\) if there is a properly embedded arc \(s\) in \(S\) such that
- \(S'\) is obtained by attaching a 1-handle to \(S\) along \(\partial s\),
- \(P'\) is defined as the union of \(P\) and the attached 1-handle,
- \(h' = R_{\sigma} \circ h\), where the extension of \(h\) to \(P'\) by the identity is also denoted by \(h\), and \(R_{\sigma}\) denotes the right-handed Dehn twist along the closed curve \(\sigma\) which is the union of \(s\) and the core of the attached 1-handle.

The effect of positively stabilizing a partial open book decomposition on the associated sutured manifold and the compatible contact structure is taking a connected sum with \((S^3, \xi_{std})\) away from the boundary. We will prove this statement in Lemma 2.11 and the notion of sutured Heegaard diagram will be helpful in our argument. So we digress to review basic definitions and properties of Heegaard diagrams of sutured manifolds (cf. [Ju]).

A sutured Heegaard diagram is given by \((\Sigma, \alpha, \beta)\), where the Heegaard surface \(\Sigma\) is a compact oriented surface with nonempty boundary and \(\alpha = \{a_1, a_2, \ldots, a_m\}\) and \(\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}\) are two sets of pairwise disjoint simple closed curves in \(\Sigma \setminus \partial \Sigma\). Every sutured Heegaard diagram \((\Sigma, \alpha, \beta)\), uniquely defines a sutured manifold \((M, \Gamma)\) as follows: Let \(M\) be the 3-manifold obtained from \(\Sigma \times [0, 1]\) by attaching 3-dimensional 2-handles along the curves \(a_i \times \{0\}\) and \(\beta_j \times \{1\}\) for \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\). The suture \(\Gamma\) on \(\partial M\) is defined by the set of curves \(\partial \Sigma \times \{1/2\}\) (see Remark 2).
In [Ju], Juhász proved that if $\left(M, \Gamma\right)$ is defined by $\left(\Sigma, \alpha, \beta\right)$, then $\left(M, \Gamma\right)$ is balanced if and only if $|\alpha| = |\beta|$, the surface $\Sigma$ has no closed components and both $\alpha$ and $\beta$ consist of curves linearly independent in $H_1(\Sigma; \mathbb{Q})$. Hence a sutured Heegaard diagram $\left(\Sigma, \alpha, \beta\right)$ is called balanced if it satisfies the conditions listed above. We will abbreviate balanced sutured Heegaard diagram as balanced diagram.

A partial open book decomposition of $\left(M, \Gamma\right)$ gives a sutured Heegaard diagram $\left(\Sigma, \alpha, \beta\right)$ of $\left(M, \Gamma\right)$ as follows: Let

$$\Sigma = P \times \{0\} \cup -S \times \{-1\}/\sim \subset \partial H$$

be the Heegaard surface. Observe that, modulo identifications,

$$\partial \Sigma = \left(\partial P \setminus A\right) \times \{0\} \cup -\left(\partial S \setminus \partial P\right) \times \{-1\} \cong -\Gamma.$$ 

As in the proof of Proposition 2.6, let $a_1, a_2, \ldots, a_r$ be properly embedded pairwise disjoint arcs in $P$ with endpoints on $A$ such that $S' \setminus \bigcup_j a_j$ deformation retracts onto $S' \setminus P$. Then define two families $\alpha := \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\beta := \{\beta_1, \beta_2, \ldots, \beta_r\}$ of simple closed curves in the Heegaard surface $\Sigma$ by

$$\alpha_j = a_j \times \{0\} \cup a_j \times \{-1\}/\sim \quad \text{and} \quad \beta_j = a_j \times \{0\} \cup h(a_j) \times \{-1\}/\sim.$$ 

$\left(\Sigma, \alpha, \beta\right)$ is a sutured Heegaard diagram of $\left(M, \Gamma\right)$. Here the suture is $-\Gamma$ since $\partial \Sigma$ is isotopic to $-\Gamma$.

**Lemma 2.11.** The balanced sutured manifold associated to a partial open book decomposition and the compatible contact structure are invariant under positive stabilization.

**Proof.** Let $\left(S, P, h\right)$ be a partial open book decomposition of $\left(M, \Gamma\right)$; $s$ be a properly embedded arc in $S$, and $\left(S', P', h'\right)$ be the corresponding positive stabilization of $\left(S, P, h\right)$. Consider the sutured Heegaard diagram $\left(\Sigma, \alpha, \beta\right)$ of $\left(M, -\Gamma\right)$ given by $\left(S, P, h\right)$ using properly embedded disjoint arcs $a_1, a_2, \ldots, a_r$ in $P$.

Let $a_0$ be the cocore of the 1-handle attached to $S$ during stabilization. The endpoints of $a_0$ are on $A' = \partial P' \cap \partial S'$ and $S' \setminus \bigcup_j a_j$ deformation retracts onto $S' \setminus P' = S \setminus P$. Using the properly embedded disjoint arcs $a_0, a_1, a_2, \ldots, a_r$ in $P'$ we get a sutured Heegaard diagram $\left(\Sigma', \alpha', \beta'\right)$ of $\left(M', -\Gamma'\right)$, where $\left(M', \Gamma'\right)$ is the sutured manifold associated to $\left(S', P', h'\right)$. Observe that $\alpha' = \{a_0\} \cup \alpha$, $\beta' = \{b_0\} \cup \beta$, and

$$\Sigma' = P' \times \{0\} \cup -S' \times \{-1\}/\sim \cong T^2 \# \Sigma.$$ 

Since $h'$ is a right-handed Dehn twist along $\sigma$ composed with the extension of $h$ which is identity on $P' \setminus P$, $\alpha_0$ is disjoint from every $\beta_j$ with $j > 0$. Therefore $\left(\Sigma', \alpha', \beta'\right)$ is a stabilization of the Heegaard diagram $\left(\Sigma, \alpha, \beta\right)$, and consequently $\left(M', \Gamma'\right) \cong \left(M, \Gamma\right)$. The contact structure $\xi'$ compatible with $\left(S', P', h'\right)$ is contactomorphc to $\xi$ since $\xi'$ is obtained from $\xi$ by taking a connected sum with $(S^3, \xi_{std})$ away from the boundary. This can be seen as in the closed case, and holds essentially because of the fact that the abstract open book with an annulus page and monodromy given by a right-handed Dehn twist (which is the one that gives the genus-1 Heegaard decomposition with a single $\alpha$-curve that intersects the single $\beta$-curve geometrically once) is compatible with the standard contact structure on $S^3$. \qed
3. Relative Giroux correspondence

The following theorem is the key to obtaining a description of a partial open book decomposition of \((M, \Gamma, \xi)\) in the sense of Honda, Kazez and Matić.

**Theorem 3.1** ([HKM09], Theorem 1.1). Let \((M, \Gamma)\) be a balanced sutured manifold and let \(\xi\) be a contact structure on \(M\) with convex boundary whose dividing set \(\partial M\) is isotopic to \(\Gamma\). Then there exist a Legendrian graph \(K \subset M\) whose endpoints lie on \(\Gamma \subset \partial M\) and a regular neighborhood \(N(K) \subset M\) of \(K\) which satisfy the following:

(A) (i) \(T = \overline{\partial N(K) \setminus \partial M}\) is a convex surface with Legendrian boundary.

(ii) For each component \(\gamma_i\) of \(\partial T\), \(\gamma_i \cap \partial M\) has two connected components.

(iii) There is a system of pairwise disjoint compressing disks \(D^2_j\) for \(N(K)\) so that \(\partial D^2_j\) is a curve on \(T\) intersecting the dividing set \(\Gamma_T\) of \(T\) at two points and each component of \((N(K) \setminus \cup_j D^2_j)\) is a standard contact 3-ball, after rounding the edges.

(B) (i) Each component \(H\) of \(\overline{M \setminus N(K)}\) is a handlebody (with convex boundary).

(ii) There is a system of pairwise disjoint compressing disks \(D^3_k\) for \(H\) so that each \(\partial D^3_k\) intersects the dividing set \(\Gamma_{\partial H}\) of \(\partial H\) at two points and \(H \setminus \cup_k D^3_k\) is a standard contact 3-ball, after rounding the edges.

Based on Theorem 3.1, Honda, Kazez and Matić describe a partial open book decomposition on \((M, \Gamma)\) in [HKM09, Section 2]. In this paper, for the sake of simplicity and without loss of generality, we will assume that \(M\) is connected. As a consequence \(M \setminus N(K)\) in Theorem 3.1 is also connected.

We claim that the description in [HKM09] gives a partial open book decomposition \((S, P, h)\); that the balanced sutured manifold associated to \((S, P, h)\) is isotopic to \((M, \Gamma)\); and that \(\xi\) is compatible with \((S, P, h)\) — all in the sense that we defined in this paper. In the rest of this section we prove these claims and Lemma 3.3 to obtain a proof of Theorem 1.1.

The tubular portion \(T\) of \(-\partial N(K)\) in Theorem 3.1(A)(i) is split by its dividing set into positive and negative regions, with respect to the orientation of \(\partial (M \setminus N(K))\). Let \(P\) be the positive region. Note that the negative region \(T \setminus P\) is diffeomorphic to \(P\). Since \((M, \Gamma)\) is assumed to be a (balanced) sutured manifold, \(\partial M\) is divided into \(R_+(\Gamma)\) and \(R_-(\Gamma)\) by the suture \(\Gamma\). Let \(R_+ = R_+(\Gamma) \setminus \cup_i \partial D_i\), where \(D_i\)'s are the components of \(\partial N(K) \cap \partial M\) and let \(S\) be the surface which is obtained from \(T_+\) by attaching the positive region \(P\). If we denote the dividing set of \(T\) by \(A = \partial P \cap \partial S\), then it is easy to see that

\[N(K) \cong (P \times [0, 1])/\sim\]

where \((x, t) \sim (x, t')\) for \(x \in A\) and \(t, t' \in [0, 1]\), such that the dividing set of \(\partial N(K)\) is given by \(\partial P \times \{1/2\}\).

In [HKM09], Honda, Kazez and Matić observed that

\[\overline{M \setminus N(K)} \cong (S \times [-1, 0])/\sim\]

where \((x, t) \sim (x, t')\) for \(x \in \partial S\) and \(t, t' \in [-1, 0]\), such that the dividing set of \(\overline{M \setminus N(K)}\) is given by \(\partial S \times \{0\}\).
Moreover the embedding \( h : P \rightarrow S \) which is obtained by first pushing \( P \) across \( N(K) \) to \( T \setminus P \subset \partial(M \setminus N(K)) \), and then following it with the identification of \( \overline{M \setminus N(K)} \) with \( (S \times [-1,0]) / \sim \) is called the monodromy map in the Honda-Kazez-Matić description of a partial open book decomposition.

In conclusion, we see that the triple \((S, P, h)\) satisfies the conditions in Definition 2.1:

1. The compact oriented surface \( S \) is connected since we assumed that \( M \) is connected and it is clear that \( \partial S \neq \emptyset \).
2. The surface \( P \) is a proper subsurface of \( S \) such that \( S \) is obtained from \( S \setminus P \) by successively attaching 1-handles by construction.
3. The monodromy map \( h : P \rightarrow S \) is an embedding such that \( h \) fixes \( A = \partial P \cap \partial S \) pointwise.

Next we observe that \( N(K) \) (resp. \( \overline{M \setminus N(K)} \)) corresponds to \( N \) (resp. \( H \)) in our construction of the balanced sutured manifold associated to a partial open book decomposition proceeding Definition 2.1. The monodromy map \( h \) amounts to describing how \( N = N(K) \) and \( H = \overline{M \setminus N(K)} \) are glued together along the appropriate subsurface of their boundaries. This proves that the balanced sutured manifold associated to \((S, P, h)\) is diffeomorphic to \((M, \Gamma)\).

**Lemma 3.2.** The contact structure \( \xi \) in Theorem 3.1 is compatible with the partial open book decomposition \((S, P, h)\) described above.

**Proof.** We have to show that the contact structure \( \xi \) in Theorem 3.1 satisfies the conditions (1), (2) and (3) stated in Proposition 2.6 with respect to the partial open book decomposition \((S, P, h)\) described above. We already observed that \( N = N(K) \) and \( H = \overline{M \setminus N(K)} \). Then

1. The restrictions of the contact structure \( \xi \) onto \( N(K) \) and \( \overline{M \setminus N(K)} \) are tight by conditions (A)(iii) and (B)(ii) of Theorem 3.1, respectively. This is because in either case one obtains a standard contact 3-ball or a disjoint union of standard contact 3-balls by cutting the manifold along a collection of compressing disks each of whose boundary geometrically intersects the dividing set exactly twice, and hence the dividing set of each of these compressing disks is a single boundary-parallel arc (see [HO02, Corollary 2.6 (2)]).

2. \( \partial H = \partial(M \setminus N(K)) = (\partial M \setminus \cup_i D_i) \cup T \) is convex by the convexity of \( \partial M \) and the convexity of \( T \) (condition (A)(i) in Theorem 3.1). Its dividing set is the union of those of \( \partial M \setminus \cup_i D_i \) and \( T \), hence it is isotopic to \( (\partial S \setminus \partial P) \times \{0\} \cup A \times \{0\} = \partial S \times \{0\} \).

3. \( \partial N = \partial N(K) = \cup_i D_i \cup T \) is convex by the convexity of \( D_i \subset \partial M \) and the convexity of \( T \). Its dividing set is the union of those of \( D_i \)'s and \( T \), hence it is isotopic to \( (\partial P \setminus \partial S) \times \{1/2\} \cup A \times \{0\} = \partial P \times \{1/2\} \). \( \square \)

The following lemma is the only remaining ingredient in the proof of Theorem 1.1.

**Lemma 3.3.** Let \((S, P, h)\) be a partial open book decomposition, \((M, \Gamma)\) be the balanced sutured manifold associated to it, and \( \xi \) be a compatible contact structure. Then \((S, P, h)\) is given by the Honda-Kazez-Matić description.

**Proof.** Consider the graph \( K \) in \( P \) that is obtained by gluing the core of each 1-handle in \( P \) (see Figure 5 for example).

It is clear that \( P \) retracts onto \( K \). We will denote \( K \times \{1/2\} \subset P \times \{1/2\} \) also by \( K \). We can first make \( P \times \{1/2\} \) convex and then Legendrian realize \( K \).
with respect to the compatible contact structure $\xi$ on $N \subset M$. This is because each component of the complement of $K$ in $P$ contains a boundary component (see [EtO, Remark 4.30]). Hence $K$ is a Legendrian graph in $(M, \xi)$ with endpoints in $\partial P \times \{1/2\} \setminus \partial S \times \{0\} \subset \Gamma \subset \partial M$ such that $N = P \times [0, 1]/\sim$ is a neighborhood $N(K)$ of $K$ in $M$. Then all the conditions except $(A)(i)$ in Theorem 3.1 on $N(K) = N$ and $M \setminus N(K) = H$ are satisfied because of the way we constructed $\xi$ in Proposition 2.6. Since $\partial N$ is convex $T$ is also convex. It remains to check that the boundary of the tubular portion $T$ of $N$ is Legendrian. Note that each component of this boundary $\partial D_i = \partial(c_i \times [0, 1]) \subset \partial N$ is identified with $\gamma_i = c_i \times \{0\} \cup h(c_i) \times \{-1\}$ in the convex surface $\partial H = S \times \{0\} \cup -S \times \{-1\}$. Since each $\gamma_i$ intersects the dividing set $\Gamma_{\partial H} = S \times \{0\}$ of $\partial H$ transversely at two points $\partial c_i \times \{0\}$, the set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is non-isolating in $\partial H$ and hence we can use the Legendrian Realization Principle to make each $\gamma_i$ Legendrian. \hfill \Box

**Proof of Theorem 1.1.** By Proposition 2.6 each partial open book decomposition is compatible with a unique compact contact 3-manifold with convex boundary up to contact isotopy. This gives a map from the set of all partial open book decompositions to the set of all compact contact 3-manifolds with convex boundary and by Remark 2 this map descends to a map from the set of isomorphism classes of all partial open book decompositions to the set of isomorphism classes of all compact contact 3-manifolds with convex boundary. Moreover by Lemma 2.11 this gives a well-defined map $\Psi$ from the isomorphism classes of all partial open book decompositions modulo positive stabilization to that of isomorphism classes of compact contact 3-manifolds with convex boundary. On the other hand, Honda-Kazez-Matić description gives a well-defined map $\Phi$ in the reverse direction by [HKM09, Theorems 1.1 and 1.2]. Furthermore, $\Psi \circ \Phi$ is identity by Lemma 3.2 and $\Phi \circ \Psi$ is identity by Lemma 3.3. \hfill \Box

### 4. Examples

Below we provide examples of abstract partial open books which correspond to some basic contact 3-manifolds with boundary. These examples were previously appeared in [EtOz] where their contact invariants were calculated.

**Example 4.1.** Let $S$ be an annulus, $P$ be a regular neighborhood of $r$ disjoint arcs connecting the two distinct boundary components of $S$ as in Figure 6, and the monodromy $h$ be the inclusion of $P$ into $S$. The partial open book $(S, P, h)$ is compatible with the contact structure obtained by removing $r$ disjoint standard
contact open 3-balls from the unique (up to isotopy) tight contact structure $\xi_{\text{std}}$ on $S^1 \times S^2$.

![Figure 6. The annulus $S$, $r$ components $P_1, \ldots, P_r$ of $P$ in Example 4.1.](image)

**Example 4.2 (Standard contact 3-ball).** Let $S$ and $P$ be as in Example 4.1 for $r = 1$, and the monodromy $h$ be the restriction (to $P$) of a right-handed Dehn twist along the core of $S$. The contact 3-manifold $(M, \Gamma, \xi)$ compatible with this partial open book is the standard contact 3-ball. Here the Legendrian graph $K$ which satisfies the conditions in Theorem 3.1 is a single arc in $B^3$ connecting two distinct points on $\Gamma$ as depicted in Figure 7. The complement $H$ of a regular neighborhood $N = N(K)$ in the standard contact 3-ball $B^3$ is a solid torus with two parallel dividing curves (see Figure 8) on $\partial H$ which are homotopically nontrivial inside $H$. Here a meridional disk in $H$ will serve as the required compressing disk $D^2$ for $H$ in Theorem 3.1 ($B$). On the other hand, $N$ is already a standard contact 3-ball. This shows in particular that the standard contact 3-ball can be obtained from a tight solid torus $H$ by attaching a tight 2-handle.

![Figure 7. The Legendrian arc $K$ in the standard contact 3-ball.](image)

**Example 4.3 (Standard neighborhood of an overtwisted disk).** Let $(S, P, h)$ be the partial open book decomposition shown in Figure 2. This is the partial open book considered in [HKM09, Example 1] which is compatible with the standard neighborhood of an overtwisted disk.

Here we observe that by Proposition 2.6, $(M, \Gamma, \xi)$ is obtained by gluing a pair of compact connected contact 3-manifolds with convex boundaries, namely $(H, \Gamma_{\partial H}, \xi|_H)$ and $(N, \Gamma_{\partial N}, \xi|_N)$, along parts of their boundaries. We know that

$$H = (S \times [-1, 0])/\sim$$
where $S$ is an annulus and $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 0]$. There is a unique (up to isotopy) compatible tight contact structure on $H$ whose dividing set $\Gamma_{\partial H}$ on $\partial H$ is $\partial S \times \{0\}$ (cf. Proposition 2.6). Hence $(H, \Gamma_{\partial H}, \xi|_H)$ is a solid torus carrying a tight contact structure where $\Gamma_{\partial H}$ consists of two parallel curves on $\partial H$ which are homotopically nontrivial in $H$. We observe that when we cut $H$ along a compressing disk we get a standard contact 3-ball $B^3$ with its connected dividing set $\Gamma_{\partial B^3}$ on its convex boundary. Note that $\Gamma_{\partial B^3}$ is obtained by “gluing” $\Gamma_{\partial H}$ and the dividing set on the compressing disk. Similarly we know that $N = (P \times [0, 1]) / \sim$, where $(x, t) \sim (x, t')$ for $x \in A$ and $t, t' \in [0, 1]$. There is a unique (up to isotopy) compatible contact structure on $N$ whose dividing set $\Gamma_{\partial N}$ on $\partial N$ is $\partial P \times \{1/2\}$ (cf. Proposition 2.6). We observe that $(N, \Gamma_{\partial N}, \xi|_N)$ is the standard contact 3-ball.

References


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