Int. J. Production Economics 228 (2020) 107692

Contents lists available at ScienceDirect

International Journal of Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Optimal control of production-inventory systems with correlated demand inter-arrival and processing times

Nima Manafzadeh Dizbin [a,*], Barış Tan [b]

[a] Graduate School of Business, Koç University, Rumeli Feneri Yolu, Istanbul, 34450, Turkey

[b] College of Administrative Sciences and Economics, College of Engineering, Koç University, Rumeli Feneri Yolu, Istanbul, 34450, Turkey

ARTICLE INFO

Keywords:
Production systems
State-dependent threshold policies
Correlated demand arrival
Correlated service process
Markovian arrival processes

ABSTRACT

We consider the production control problem of a production-inventory system with correlated demand inter-arrival and processing times that are modeled as Markovian Arrival Processes. The control problem is minimizing the expected average cost of the system in the steady-state by controlling when to produce an available part. We prove that the optimal control policy is the state-dependent threshold policy. We evaluate the performance of the system controlled by the state-dependent threshold policy by using the Matrix Geometric method. We determine the optimal threshold levels of the system by using policy iteration. We then investigate how the autocorrelation of the arrival and service processes impact the performance of the system. Finally, we compare the performance of the optimal policy with 3 benchmark policies: a state-dependent policy that uses the distribution of the inter-event times but assumes i.i.d.inter-event times, a single-threshold policy that uses both the distribution and also the autocorrelation, and a single-threshold policy that uses the distribution of the inter-event times but assumes they are not correlated. Our analysis demonstrates that ignoring autocorrelation in setting the parameters of the production policy causes significant errors in the expected inventory and backlog costs. A single-threshold policy that sets the threshold based on the distribution and also the autocorrelation performs satisfactorily for systems with negative autocorrelation. However, ignoring positive correlation yields high errors for the total cost. Our study shows that an effective production control policy must take correlations in service and demand processes into account.

1. Introduction

Controlling production systems to match supply and demand in an uncertain environment received considerable attention in the manufacturing systems literature. Control policies such as the Control-Point Policy, Generalized Kanban Policy, and Base-Stock Policy are suggested to control the material flow in a production system, e.g., Gershwin (2000), Duri et al. (2000) and Liberopoulos and Dallery (2000) among others.

The analytical models that evaluate the performance of production systems controlled to match supply and demand usually model the demand inter-arrival and processing times as independent random variables. As a result, dependence among the inter-arrival and processing times is not often taken into account. However, autocorrelation can be observed in processing, inter-arrival and inter-departure times. Fig. 1 depicts the empirical distributions and autocorrelations of the processing, inter-arrival, and inter-departure times of certain equipment at the Robert Bosch semiconductor manufacturing plant. Schomig and Mittler (1995) and Inman (1999) also report dependence in observed cycle and inter-event times. The simulation and analytical studies also show negative dependence among the inter-departure times of the products leaving a production line (Hendricks and McClain, 1993; Tan and Lagershausen, 2017; Manafzadeh Dizbin and Tan, 2019).

Complicated processing tasks such as batch processing, parallel processing, and merging may create high dependence between the processing times of the products. Moreover, dispatching rules and the production network for different products yield dependence among the inter-arrival times observed at different stations. Correlated inter-arrival and processing times then result in a correlated output process. The correlated output process creates the arrival process at other stages of the production and causes further dependence among inter-departure times.

Ignoring dependence among inter-event times has been one of the shortcoming of the classical queuing theory in analyzing manufacturing systems (Shanthikumar et al., 2007). Although the optimal inventory control policies with independent and identically distributed (i.i.d.) inter-event times have been investigated thoroughly in the literature,

* Corresponding author.
E-mail address: ndizbin14@ku.edu.tr (N. Manafzadeh Dizbin).

https://doi.org/10.1016/j.ijpe.2020.107692
Received 10 January 2019; Received in revised form 30 December 2019; Accepted 19 February 2020
Available online 24 February 2020

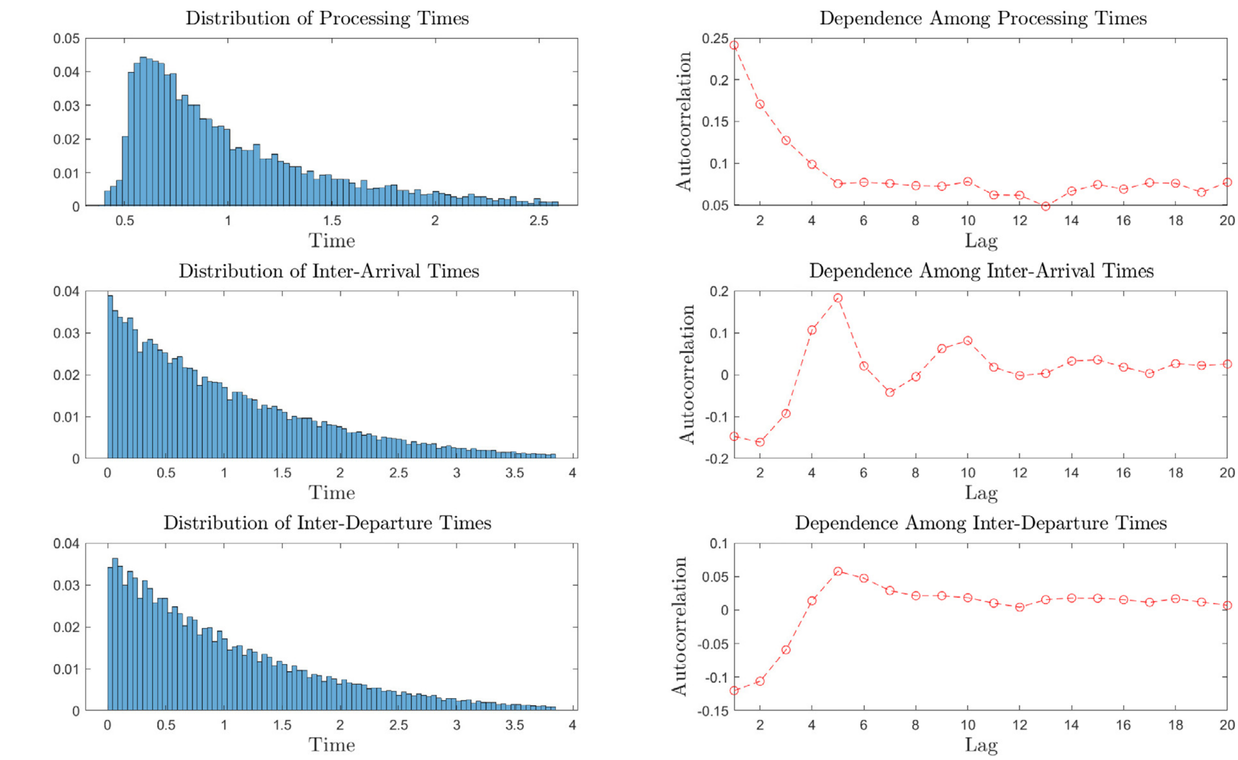

Fig. 1. Empirical distribution and dependence of the processing, inter-arrival, and inter-departure times of a specific equipment at a semiconductor manufacturing plant.

the optimal production control policies of production systems with correlated inter-event times have not been studied. The objective of this paper is to fill this gap by deriving the optimal control policy of a manufacturing system with correlated inter-arrival and processing times and analyzing the effects of correlation on the production control.

The control problem studied in this paper is minimizing the expected holding and backlog costs of a production-inventory system with correlated processing and demand inter-arrival times that are modeled as Markovian Arrival Processes (MAP) in the long run. The action space consists of whether or not to produce depending on the state of the system. We prove that a manufacturing system with MAP demand inter-arrivals and MAP processing times can be controlled optimally by using a state-dependent threshold policy. We use a matrix-geometric method to evaluate the performance of a production-inventory system controlled by the state-dependent threshold policy. We determine the optimal threshold levels by using a policy iteration method. We then evaluate the impact of positive and negative autocorrelations in inter-arrival and processing times. In addition, we compare the performance of the optimal policy in controlling a system with 3 benchmark production policies: a state-dependent policy that uses the distribution but assumes i.i.d. inter-event times, a single-threshold policy that uses both the distribution and also the autocorrelation, and a single-threshold policy that uses the distribution but assumes i.i.d. inter-event times.

We consider proving the optimal control policy of a production-inventory system with correlated inter-arrival and processing times that are modeled as MAPs, proposing the Matrix Geometric method to evaluate the performance of the system when it is controlled optimally, and investigating the impact of autocorrelation among the inter-event times on the performance of a production-inventory system as the main contributions of this paper.

The structure of the paper is as follows. The literature related to the optimal control of production and queuing-inventory systems is reviewed in Section 2. In Section 3, the problem is defined and it is proven that the state-dependent threshold policy is the optimal control policy. The Matrix Geometric method to evaluate the performance of the system for given thresholds and the policy iteration approach used to determine the optimal thresholds are presented in Section 4. Section 5 evaluates the impact of positive and negative autocorrelation in demand inter-arrival and processing times on the performance of the state-dependent threshold policy and compares the performance of the benchmark policies with the optimal policy. Finally, Section 6 concludes the paper and discusses future research directions.

2. Literature review

We discuss the pertinent literature in three related areas. The first area is related to the papers that investigate the optimal control policy of inventory systems with correlated demand-arrivals where the demand is modeled as a Markov-modulated process. The second area is related to the papers that study the optimal control of production systems. Finally, the third area is related to the papers that evaluate the performance of queuing-inventory systems by using matrix-geometric methods.

2.1. Optimal control of inventory systems

In this section, we cover the inventory control literature that uses models where the demand is modeled as Markov-modulated (MM) processes. The optimal inventory control policy for these models is proven to be state-dependent base-stock or (s, S) policies under different assumptions about the cost criterion, lead time, timing horizon, and production capacity. Accordingly, an order of an appropriate quantity is triggered when the inventory position is below a given threshold.

Song and Zipkin (1993) investigate the optimality of state-dependent base-stock and (s, S) policies for an inventory system under continuous review with Markov-modulated Poisson Process (MMPP) demand process, and fixed ordering costs over finite and infinite horizon. They show that the state-dependent base-stock policy is optimal when there is no fixed cost and the state-dependent (s, S) policy is optimal when there is a fixed cost. Song and Zipkin (1996a) evaluate the steady-state performance measures of multi-echelon systems with MMPP demand and state-independent base-stock policies in multi-echelon inventory systems. Song and Zipkin (1996b) evaluate the performance measures of a two-echelon inventory system with MMPP demand where the first (second) stage is controlled by a state-independent (dependent) base-stock policy. Bayraktar and Ludkovski (2010) consider a continuous-time model for inventory management with Markov-modulated non-stationary demand where the state of the modulating processes is unobserved. They prove the optimality of a time-and-belief-dependent (s, S) strategy and develop a numerical method to calculate the optimal policy. Nasr and Maddah (2015) consider a continuous inventory replenishment system with a MMPP demand process.

Sethi and Cheng (1997) consider a discrete-time model with non-stationary demand and no lead time and investigate the optimality of (s, S) policies for finite and infinite horizon and non-stationary systems. Beyer and Sethi (1997) consider a system with convex surplus cost and prove that the (s, S) model with MMPP demand and average cost criterion minimizes the inventory cost. Özekici and Parlar (1999) consider a model involving random variations in supply and prove the optimality of the state-dependent base-stock policies and state-dependent (s, S) type policies with and without fixed setup costs. The optimality of the (s, S) policy is generalized to systems involving general costs (Beyer et al., 1998), and lost sales (Cheng and Sethi, 1999). Chen and Song (2001) show the optimality of the state dependent base-stock policies for serial systems with Markov modulated demand and deterministic lead time under a finite and infinite horizon and average cost criterion. Muharremoglu and Tsitsiklis (2008) study a multi-echelon inventory system with Markov-modulated demand under periodic review and propose a single item-single customer approach to prove the optimality of state-dependent base-stock policies. Janakiraman and Muckstadt (2009) utilize the method developed in Muharremoglu and Tsitsiklis (2008) to demonstrate the optimality of the state dependent base-stock policies in a two-echelon serial system with identical ordering/production capacities. Hu et al. (2016) study a class of periodic review (s, S) inventory systems with a Markov-modulated demand process. They develop an algorithm to calculate the moments of the inventory level.

The main objective in these studies is determining the amount of inventory to be shipped to the next level of supply chain where the inventory is supplied exogenously without any capacity restrictions. One of the main assumptions of the infinite-capacity problems is the independence between lead times of the products shipped between two consecutive echelons of the supply chain. As a result, orders given at different times can pass each other. This property is crucial in proving the optimal policy. Hence, the researchers that study the finite capacity problems in an inventory control setting usually model the lead time as deterministic or consider a finite time horizon to deal with this problem. The main difference in our study is considering the optimal production control problem of a finite-capacity producer that has correlated processing times and meets a demand stream with correlated inter-arrival times.

2.2. Optimal control of production systems

The second stream of the literature is related to the studies that consider the optimal control of production systems. These studies use models with discrete or continuous flow of materials. It is shown that threshold-type policies that are referred as the *base-stock* policies for the models with discrete material flow and the *hedging*-type policies for the models with continuous material flow are optimal. Accordingly, production is allowed when the inventory level is below a threshold determined for the given state.

For a system with discrete material flow, Veatch and Wein (1994) study a make-to-stock manufacturing system with an exogenous Poisson demand and two stations. Each station is modeled as a queue with controllable production rate and exponential service times. The objective of the study is to control the production rates to minimize the inventory holding and backordering costs. Berman and Kim (2001) consider the dynamic replenishment of parts in a supply chain with single class of customers where parts are procured by a supplier with an Erlang processing distribution. They assume Poisson customer arrivals and exponential processing times and model the problem as a Markov decision process. They show that the optimal ordering policy that minimizes the customer waiting, inventory holding, and order replenishment costs has a monotonic threshold structure. He et al. (2002) examine several inventory replenishment policies for a make-to-order production-inventory system with Poisson demand-arrival. They derive the optimal replenishment policy, which minimizes the average total cost per product of the warehouse. de Véricourt et al. (2002) consider a capacitated supply system with a single product and several classes of customers where each customer class has a different backorder cost. They study the optimal allocation policy of products and show the optimality of a threshold policy. Karabağ and Tan (2019) analyze the purchasing, production, and sales policies for a continuous-review discrete material production-inventory system with exponentially distributed demand inter-arrival, and processing times. They show that the optimal purchasing, production, and sales strategies are state-dependent threshold policies.

For a system with continuous material flow and constant demand for a single product, Sharifnia (1988) study the production control of a manufacturing system with arbitrary number of machine states. He shows that the optimal production policy that minimizes the average inventory and backlog costs of the system is the hedging-point policy. Tan (2002) considers a manufacturing system with two-state Markov modulated demand, uncertain repair and failure times and continuous material. He shows that the optimal production flow control policy that minimizes the expected average inventory holding and backlog costs is a double-hedging policy. Gurkan et al. (2007) use simulation-based optimization to determine the threshold levels of a production-inventory system where stochasticity in the system is modeled using semi-Markov processes. Gershwin et al. (2009) consider a manufacturing system with deterministic production time and stochastic Markov modulated demand. They show that the hedging point is the optimal control policy of the system. Tan (2018) study the optimal production flow control problem of a make-to-stock manufacturing system with price, cost, and demand uncertainty. He models the stochastic dynamics of the system with a time-homogeneous Markov chains and shows that the optimal production policy is a state-dependent hedging policy.

All of these models assume deterministic or i.i.d. demand inter-arrival and service times and do not consider correlation explicitly. Most of the studies further assume that the inter-event times are exponentially distributed random variables. Our study differs from the production control literature in that we analyze a discrete material-flow continuous-time production-inventory system with correlated demand inter-arrival and service times modeled as Markovian Arrival Processes.

2.3. Matrix-geometric methods

We now focus on the literature that utilize the matrix-geometric methods for performance evaluation of the queuing-inventory system. He and Jewkes (2000) use the matrix-analytic methods to evaluate the performance of a make-to-order production-inventory system with Poisson arrivals and exponential processing times. Manuel et al. (2007, 2008) study a perishable (s, S) inventory system under continuous review with a finite buffer and a single server. They consider two types of customers arrivals modeled as a MAP and service process with phase-type distribution. They evaluate the joint probability distributions of the number of customers in the system and the inventory level in the steady state. Zhao and Lian (2011) consider a queuing-inventory system with Poisson demand arrival, exponential processing times and (r, Q) replenishment policy. Their objective is minimizing the long-run expected waiting cost. They formulate the problem as a level-dependent Quasi-Birth-and-Death process and investigate the control of such a process. Liu et al. (2014) investigate a Markovian inventory system with two classes of demands, replenishment policy, and a flexible service discipline. They derive the steady state probability distribution of the inventory levels by using Markovian processes, and adopt a mix integer optimization model to find the optimal inventory control levels. Jiang et al. (2015) consider a two-echelon queuing-inventory system with a demand process that follows a compound Poisson process, a two-echelon inventory system consisting of a central warehouse and several sub-warehouses. They propose an algorithm for minimizing the mean total cost of the inventory system. Xia et al. (2017) study the service rate control problem of the MAP/M/1 queue. They evaluate the impact of service rates on the long-run average total cost of the system and show the optimality of quasi-threshold-type policy under some conditions. Manafzadeh Dizbin and Tan (2019) study a production-inventory control system with correlated service and processing times controlled with a single base-stock level that is not the optimal policy for this system. They evaluate the performance of the models with the Matrix Geometric method and evaluate the effect of autocorrelation on the performance when the system is controlled with this sub-optimal policy.

These studies that use the matrix-geometrics methods in the literature focus on the performance evaluation of queuing-inventory systems under a given production policy where arrival and service processes are modeled usually as independent distributions. Our study differs from this stream of the literature in that we derive the optimal control policy and use the Matrix Geometric method to determine the performance measures of a system with correlated demand-arrival and processing times.

3. Model

We consider a single machine with an unlimited buffer where the raw material is supplied from an unlimited stock with zero lead-time. An arriving demand to the system is satisfied immediately, if there is enough inventory in the buffer to meet the demand. Otherwise, the demand is backlogged. Since backlog is allowed, all demand is satisfied eventually according to the first-come-first-served (FIFO) rule. The difference between the cumulative production and demand is referred as the inventory position and denoted by $X(t)$. The on-hand inventory is $X^+(t) = \max\{X(t), 0\}$ and the backlogged demand level is $X^-(t) = \max\{-X(t), 0\}$.

The cost structure of the system consists of the holding and backlog costs. The holding cost is h per unit per unit of time, and the backlog cost is b per unit per unit of time. The cost function at any time t is a function of $X(t)$ and given as

$$C(X(t)) = \begin{cases} bX^-(t), & \text{if } X(t) < 0, \\ hX^+(t), & \text{if } X(t) \geq 0, \end{cases} \tag{1}$$

The demand inter-arrival times and processing times are modeled as discrete state-space and continuous-time processes. The state of the demand arrival process is denoted with $j_a \in J_a$. The state of the production time process, also referred as the service process, is denoted with $j_s \in J_s$. At any given time, the demand arrival process can be in one of $m_a = |J_a|$ discrete states and the service process can be in one of $m_s = |J_s|$ states. When the machine is available, it may or may not start producing a new part depending on the control policy.

We assume that the system is continuously reviewed, and the state of the system is fully observed at any time t. The state of the system is fully specified by the inventory position and the states of the demand arrival and service processes.

3.1. Production control problem

The main goal of the production control problem is to determine the production policy Π that minimizes the long run average cost of the system in the steady state by deciding on whether to produce or continue to producing ($u = 1$) or not ($u = 0$) depending on the state of the system (X, j_a, j_s). The average cost of this system in steady state under policy Π can be written as Eq. (2).

$$V^{\Pi}(X, j_a, j_s) = E^{\Pi}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T C\left(X(t)|X(0) = 0, J(0) = (j_a^0, j_s^0)\right) dt\right]. \tag{2}$$

Then the objective of the problem is to identify the optimal policy Π^* defined as

$$\Pi^* = \sup_{\Pi} V^{\Pi}(X, j_a, j_s). \tag{3}$$

3.2. Demand inter-arrival and processing times

In order to capture the autocorrelation in inter-event times, the demand inter-arrival and processing time processes are modeled as Markov Arrival Processes. MAPs are generalization of phase-type (Ph) distributions (Neuts, 1979). MAPs contain most of the commonly used arrival processes such as Erlang processes, Coxian distributions and Markov-modulated Poisson processes as subclasses. MAPs can approximate a given inter-event process arbitrarily close enough (Asmussen and Koole, 1993). MAPs are used commonly in the telecommunications literature to model positively correlated inter-arrivals (Buchholz et al., 2014). MAPs can also be used in modeling negatively correlated arrivals in manufacturing systems (Hendricks and McClain, 1993; Manafzadeh Dizbin and Tan, 2019). Manafzadeh Dizbin and Tan (2019) review various algorithms to construct a MAP by using the observed inter-event time data.

A MAP consists of two different sub-processes each of which has a discrete state space referred as phases. The MAP of the demand inter-arrival time is denoted as MAP(D_0, D_1). The non-diagonal elements of matrix D_0 include the transition rates between the phases that do not generate a demand arrival. The diagonal elements of the D_0 correspond to the rates of the exponentially distributed sojourn times in corresponding states. The elements of D_1 capture the transition rates that generate a demand arrival.

A MAP can be interpreted as a continuous-time Markov-chain with the generator matrix $D = D_0 + D_1$, and $|D|$ states. The joint probability density function of the consecutive inter-arrival times T_i, $i = 0, 1, \ldots$, of the MAP(D_0, D_1) is written as:

$$\begin{aligned} &f(t_1, t_2, \ldots, t_k) \\ &\quad = \beta \mathbf{exp}^{D_0 t_1} D_1 \mathbf{exp}^{D_0 t_2} D_1 \ldots \mathbf{exp}^{D_0 t_k} D_1 \mathbb{1}, \text{ for } t_i \geq 0, i \in \{1, \ldots, k\} \end{aligned} \tag{4}$$

where β is the solution of $\beta(-D_0)^{-1} D_1 = \beta$ and $\beta\mathbb{1} = 1$ and $\mathbb{1}$ is a vector of ones with an appropriate size. β can be interpreted as the probability distribution of the phases immediately after an arrival.

The nth moment of T, $E[T^n]$ is calculated from the matrix D_0 as:

$$E[T^n] = n!\beta\left(-D_0\right)^{-n}\mathbb{1}. \tag{5}$$

Accordingly, the expected value and the variance of T, $E[T]$ and $Var[T]$ are

$$\begin{aligned} E[T] &= -\beta\left(D_0\right)\mathbb{1}, \\ Var(T) &= 2\beta D_0^{-2}\mathbb{1} - \left(\beta D_0^{-1}\mathbb{1}\right)^2. \end{aligned}$$

The squared coefficient of variation of the inter-event times is $scv = Var(T)/E^2[T]$. The covariance of T_0 and T_k, Cov$\left(T_0, T_k\right)$ is also determined from the matrices D_0 and D_1 as

$$\text{Cov}\left(T_0, T_k\right) = \mathbf{E}\left(T_0 T_k\right) - \mathbf{E}(T)^2$$

$$= \mathbf{E}(T)\frac{\beta\left(-\boldsymbol{D}_0\right)^{-1}}{\beta\left(-\boldsymbol{D}_0\right)^{-1}\mathbb{1}}\left((-\boldsymbol{D}_0)^{-1}\boldsymbol{D}_1\right)^k\left(-\boldsymbol{D}_0\right)^{-1}\mathbb{1} - \mathbf{E}(T)^2. \quad (6)$$

The kth-lag autocorrelation coefficient ρ_k is defined as

$$\rho_k = \frac{Cov(T_0, T_k)}{Var(T)}. \quad (7)$$

The production time is denoted with T_s with mean $E[T_s]$, variance $Var[T_s]$ and the squared coefficient of scv_s. The MAP of the production time process is denoted as MAP(A_0, A_1) and the matrices A_0 and A_1 are defined similar to the definition of D_0 and D_1. Let the infinitesimal generator of the underlying Markov chain matrix of the production process be $A = A_0 + A_1$. Then, the state transition of the system is governed by the underlying Markov chain that has an infinitesimal generator $M = D \oplus A$ ($\oplus$ is the Kronecker sum) with a finite state space $J = J_a \times J_s$. The underlying Markov chain consists of $m = |M| = m_a \times m_s$ states and the state of the underlying process is (j_a, j_s).

3.3. Structure of the optimal control policy

In order to identify the optimal control policy that determines whether to produce or continue to produce or not depending on the state of the system (X, j_a, j_s), we discretize the continuous-time Markov process by using the uniformization technique and write the Bellman optimality equation of this system as

$$\begin{aligned} V(X, j_a, j_s) + g = {} & \frac{C(X)}{\alpha} \\ & + \sum_{j \in J_a} P_0(j_a, j)V(X, j, j_s) + \sum_{j \in J_a} P_1(j_a, j)V(X-1, j, j_s) + \\ \min & \begin{cases} 0 & u = 0 \\ \displaystyle\sum_{j' \in J_s} R_0(j_s, j')V(X, j_a, j') + \sum_{j' \in J_s} R_1(j_s, j')V(X+1, j_a, j') \\ \quad -V(X, j_a, j_s) & u = 1, \end{cases} \end{aligned} \quad (8)$$

where g is the optimal average cost of the system over an infinite horizon and $V(X, j_a, j_s)$ is the differential cost of being in state (X, j_a, j_s), the matrices P_0, R_0, P_1, and R_1 capture transitions related to events without and with the demand-arrival and service completion. The probability transition matrices of the arrival and service processes are written as

$$\begin{aligned} P_0 &= \frac{1}{\alpha}D_0 + I_{m_a}, \\ P_1 &= \frac{1}{\alpha}D_1, \\ R_0 &= \frac{1}{\alpha}A_0 + I_{m_s}, \\ R_1 &= \frac{1}{\alpha}A_1 \end{aligned} \quad (9)$$

where $\alpha \geq \max_i(-M(i,i))$ is the uniformization constant, I_{m_a} and I_{m_s} are identity matrices of the size m_a and m_s, respectively.

Our main result given in Theorem 1 proves that the state-dependent threshold policy is the optimal policy to control this system.

Theorem 1. *A state-dependent threshold policy is the optimal control policy of a production system with correlated demand-arrival and processing times modeled as MAP. According to the state-dependent threshold policy, described with the binary variable $u(x, j_a, j_s)$, the state of the modulating Markov process (j_a, j_s) and the inventory level determine whether to start, stop, or continue the production. That is, when the optimal policy is used, the production starts or continues ($u(x, j_a, j_s) = 1$) if the state of the modulating Markov process is (j_a, j_s) and the inventory level (X) is less than the threshold-level associated with the state (j_a, j_s) denoted as $Z(j_a, j_s)$. Similarly, the production stops ($u(x, j_a, j_s) = 0$) if the inventory level is greater than or equal to $Z(j_a, j_s)$. The policy can be stated as follows:*

$$u(x, j_a, j_s) = \begin{cases} 1, & \text{if } x < Z(j_a, j_s), X = x \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Proof. Given in Appendix.

Let the states of the Markov-modulating process be indexed from 1 to m. The state-dependent threshold policy is determined by the vector of the thresholds denoted by $\bar{Z} = (Z^{(m)}, Z^{(m-1)}, \ldots, Z^{(2)}, Z^{(1)})$ where $Z^{(i)}$ is the $(m - i + 1)$th biggest threshold level. The states of the Markov-modulating process are ordered according to the ordering of $\bar{Z}$. Since the steady-state performance of the system depends on the selection of these thresholds, the optimal policy is determined by selecting the optimal thresholds.

Fig. 2 shows the evolution of the inventory position, the shortfall, the state of the arrival process, and the control policy of a production-inventory system with a two-state MAP inter-arrival and exponential service times controlled by the state-dependent threshold policy. The threshold levels corresponding to state 2 and 1 are set to $(Z^{(2)}, Z^{(1)}) = (10, 5)$. Depending on the state of the arrival process, we stop the production if the inventory position is equal to or above the corresponding threshold level.

4. A matrix geometric approach to evaluate the steady-state performance of the system

In this section, we present the Matrix Geometric approach to evaluate the performance of a system controlled by the optimal state-dependent threshold policy given in Theorem 1. We first present a method to generate the infinitesimal generator matrix of a production system controlled with the given thresholds of the state-dependent threshold policy, denoted by Q. We then discuss how to determine the steady-state probabilities by using the Matrix Geometric approach.

4.1. The steady-state probability distribution

In order to capture the dynamics of the system, we focus on the shortfall from the largest threshold. The shortfall level k is equivalent to the inventory position $Z^{(m)} - k$ as shown in Fig. 2 for the specific case. The steady-state probability of being in state $(Z^{(m)} - k, j_a, j_s)$ is denoted as $\pi(k, j_a, j_s)$. The steady-state probability vector $\pi_k = \{\pi(k, j_a, j_s)\}$, $j_a \in J_a$, $j_s \in J_s$ contains the probabilities at the shortfall level k. The probabilities in π_k are ordered according to the ordering of states in the underlying Markov chain of the system.

The steady-state probability distribution is given by the vector $\pi = \{\pi_k\}$, $k = 0, 1, \ldots$. The steady-state probabilities satisfy $\pi Q = 0$ and $\pi\mathbb{1} = 1$. Therefore, once the infinitesimal generator matrix of the system is constructed, the steady-state probabilities can be calculated, and the performance of the system can be evaluated from the steady-state probability distribution.

4.2. Infinitesimal generator matrix

The generator matrix Q is determined by the submatrices that are associated with three types of events: demand arrival, production completion, or an event that does not change the inventory position for each level of the inventory position.

The submatrix that captures transitions associated with demand arrivals is referred as, the forward matrix. The state transition rates of the arrival and production process without a demand arrival or service completion are captured in the local matrix. Finally, the backward matrix captures the transition rates related to service completion. The forward, local, and backward matrices at the shortfall level k are denoted by F_k, L_k, and B_k respectively. Accordingly, the infinitesimal generator matrix has the following block tri-diagonal form:

$$Q = \begin{pmatrix} L_0 & F_0 & & & & & \\ B_1 & L_1 & F_1 & & & & \\ & B_2 & L_2 & F_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_k & L_k & F_k & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (11)$$

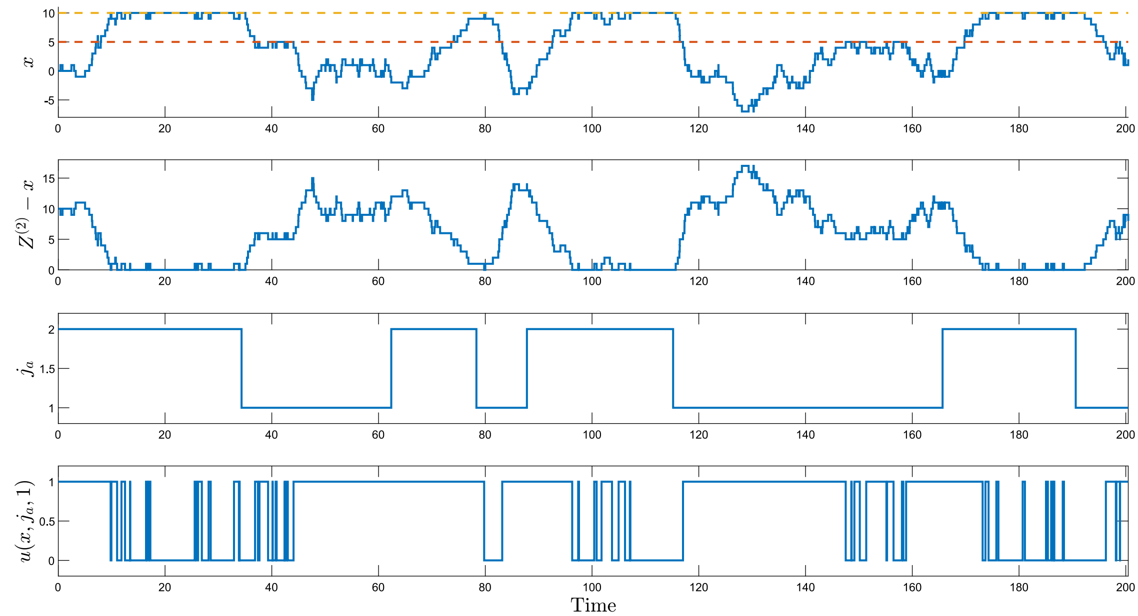

Fig. 2. Production-Inventory System with MAP Inter-Arrivals with Two States and Exponential Service Times Controlled by the State-Dependent Threshold Policy ($(Z^{(2)}, Z^{(1)}) = (10, 5)$, $E[T] = 1.1250$, $scv = 1.5$, $\rho_1 = 0.15$, $E[T_s] = 1$).

The submatrices F_k, L_k, and B_k are determined by the matrices that describe the demand arrival and service MAPs and also by the production policy.

The transition rates related to demand arrivals are given in the D_1 matrix of the demand arrival MAP. Therefore, the forward matrix F_k that includes the transitions that increases the shortfall (decreases the inventory position) by one unit is completely determined by the D_1 matrix:

$$F_k = D_1 \otimes I_{m_s}. \tag{12}$$

On the other hand, the shortfall decreases by one unit after a production completion. The production decision is authorized by the production policy at each decision epoch. The state-dependent threshold policy described in Eq. (10) makes production authorization decisions based on the inventory position and the threshold levels. Let U_k be a diagonal matrix of size $m \times m$ at the shortfall level k that authorizes the production for each state. Let $m, \ldots, 1$ be the states of the underlying Markov process corresponding to states with the threshold levels of $Z^{(m)}, \ldots, Z^{(1)}$. U_k is defined as

$$U_k = \begin{cases} U^{(i+1)}, & \text{if } Z^{(m)} - Z^{(i)} \geq k > Z^{(m)} - Z^{(i+1)}, i = 1, \ldots, m-1 \\ U^{(1)}, & \text{if } k > Z^{(m)} - Z^{(1)} \end{cases} \tag{13}$$

where $U^{(i)}$ is a diagonal matrix of the size U_k. The (j, j) element of $U^{(i)}$ corresponds to the control decision of state $m - j + 1$ as given below:

$$U^{(i)}(j, j) = \begin{cases} 1, & \text{if } \ i \leq m - j + 1 \leq m \\ 0, & \text{if } 1 \leq m - j + 1 \leq i - 1, \end{cases} \quad j = 1, \ldots, m. \tag{14}$$

The backward matrix is determined by A_1 matrix of the production time matrix and also the production authorization matrix U_k as

$$B_k = U_k \left(I_{m_a} \otimes A_1 \right). \tag{15}$$

Similarly, the local matrices for the transition rates of the events that do not change the inventory position are given as

$$L_k = D_0 \otimes I_{m_s} + U_k \left(I_{m_a} \otimes A_0 \right), \tag{16}$$

$$L_0 = D_0 \otimes I_{m_s}. \tag{17}$$

The production authorization matrix appears in the definition of the local matrix L_k given in Eq. (16) in order to make the transition matrix a valid infinitesimal generator and ensure that the transition rates are zero in the states where production is not authorized.

4.3. Determining the performance measures

For given thresholds, the production policy imposed by these thresholds is implemented by defining the production authorization matrix U_k in Eq. (13). Then the generator matrix Q for the given thresholds is constructed by determining the forward, backward, and local matrices as given in Eqs. (12), (15), (16), and (17).

In order to compute the steady-state probability distribution, the inventory position that can take values between $Z^{(m)}$ and $-\infty$ can be truncated at a lower bound K^*. In this case, the size of the generator matrix will be $(Z^{(m)} + K^*)m \times (Z^{(m)} + K^*)m$. In principle, the steady-state probabilities π can be determined by solving $\pi Q = 0$ and $\pi \mathbb{1} = 1$. However, this approach can be computationally demanding if $(Z^{(m)} + K^*)m$ is large.

Alternatively, the tri-diagonal structure of the generator matrix can be exploited to determine π more efficiently. Equations $\pi Q = 0$ and $\pi \mathbb{1} = 1$ can be rewritten by using the tri-diagonal structure. Accordingly, the steady-state probabilities π_k satisfy the following equations:

$$\pi_0 L_0 + \pi_1 B_1 = 0 \tag{18}$$

$$\pi_k F_k + \pi_{k+1} L_{k+1} + \pi_{k+2} B_{k+2} = 0, \quad k = 0, 1, \ldots \tag{19}$$

$$\sum_{x=0}^{\infty} \pi_k = 1 \tag{20}$$

The equations for the forward, backward, and local matrices given in Eqs. (12), (15), (16), and (17) show that the forward matrix remains constant at each level. However, the backward and local matrices differ depending on the inventory position and the state of the underlying Markov chain. This structure is equivalent to the structure of a level-dependent Quasi Birth and Death (LDQBD) process.

There are efficient methods developed for the analysis of LDQBD processes. We determine the steady-state probabilities of this process by using the Matrix Geometric method developed for LDQBD processes (Bright and Taylor, 1995; Baumann and Sandmann, 2010).

Once the steady-state probabilities are obtained, the performance measures of the system are then calculated by using the steady-state probability distribution for the given threshold vector $\bar{Z}$ used in the threshold-based production policy. As functions of $\bar{Z}$, the expected inventory level, $X^{+}(\bar{Z})$, the expected backlog level, $X^{-}(\bar{Z})$, the probability of not having inventory in the system, $P_0(\bar{Z})$, and the total cost, $TC(\bar{Z})$ are given as:

$$\begin{aligned} X^{+}(\bar{Z}) &= \sum_{k=0}^{Z^{(m)}-1} (Z^{(m)}-k)\pi_k \mathbb{1}, \\ X^{-}(\bar{Z}) &= \sum_{k=Z^{(m)}+1}^{\infty} (k-Z^{(m)})\pi_k \mathbb{1}, \\ P_0(\bar{Z}) &= \sum_{x=Z^{(1)}+1}^{\infty} \pi_k \mathbb{1}, \\ TC(\bar{Z}) &= bX^{-}(\bar{Z}) + hX^{+}(\bar{Z}). \end{aligned} \tag{21}$$

4.4. Determining the optimal state-dependent threshold levels

The performance measures given in Eq. (21) are obtained for *given* thresholds. In order to determine the *optimal* thresholds that minimize the total cost, an effective search method is needed. We use the policy iteration algorithm (Bertsekas, 2005) to determine the optimal threshold levels of the state-dependent threshold policy. The policy iteration works with a finite Markov Decision Process (MDP) and a finite number of policies, and guarantees that an optimal policy and optimal value function are determined in a finite number of iterations.

We generate the initial MDP by truncating the inventory level between a maximum threshold level of the state-dependent threshold policy and a lower level of the inventory position. An appropriate level of the lower bound K^* is set to accumulate the probabilities of being in levels lower than K^* into level K^* by using the approach given in Heindl et al. (2004). Then at each iteration, the transition probabilities of the system are determined by using the block matrices given in Eq. (11). Once the finite-size generator matrix is determined in this way, at each iteration of the algorithm, the steady-state probabilities and the total cost are calculated by using the methodology given in Section 4.3 for the given thresholds. The algorithm is initialized by setting the state-dependent threshold levels equal to the optimal single-threshold level. The new policies are generated by implementing policy improvement. This results in increasing or decreasing the state-dependent threshold levels compared to the previous iteration. The policy improvement iterations are continued until the cost cannot be improved further.

The steady-state probabilities of the system controlled by the optimal policy obtained from the policy iteration are calculated and the performance measures for the system controlled with the optimal thresholds are obtained by using Eq. (21) with these steady-state probabilities.

5. Numerical experiments

In this section, we analyze the effect of autocorrelation in inter-arrival and processing times on the performance of a production system controlled by the state-dependent threshold policy. We consider processes with positive and negative first-lag autocorrelations. Our objectives in these experiments are two-fold: investigating the impact of positive and negative autocorrelation on the optimal performance and also comparing the optimal performance with the benchmark production policies.

5.1. Experimental setup

5.1.1. Benchmark production control policies

We use three sub-optimal production control policies as benchmarks to compare the performance of the optimal policy. The first benchmark policy, referred as Multiple-Threshold No Autocorrelation (MTNA), uses a state-dependent threshold policy where the thresholds are determined by considering the inter-event time distributions but assuming that the inter-event times are not correlated. In order to determine the state-dependent threshold for this case, the correlated inter-event time is modeled with a MAP with zero autocorrelation (MAP$^{\mathrm{ren}}$). The thresholds are determined by using the methodology in Section 4.4 and the steady-state probabilities are obtained for the system controlled with these thresholds.

The second approximation, referred as Single-Threshold With Autocorrelation (STWA), uses a single-threshold policy where the single-threshold is set optimally considering the inter-event distribution and the autocorrelation to represent the inter-event time

Finally, the third approximation, referred as Single-Threshold No Autocorrelation (STNA), uses a single-threshold policy where the optimal value of the single-threshold is determined by considering the inter-event distribution but assuming that the inter-event times are not correlated.

These different benchmark policies yield different threshold vectors to be used by the production policy. The total costs, the expected inventory levels, the expected backlog levels, and the probability of not having inventory in the system are determined for each benchmark case by evaluating the original system with the correlated inter-event times when the system is controlled with the thresholds given by the benchmark cases. These performance measures are compared with the performance measures obtained by using the optimal state-dependent threshold policy (Multiple-Threshold With Autocorrelation). The percentage deviation of the expected inventory, expected backlog, probability of having no on-hand inventory, and the total cost with a benchmark policy with respect to these measures obtained with the optimal policy are denoted with Δ_{X^+}, Δ_{X^-}, Δ_{P_0}, and Δ_{TC} respectively.

5.1.2. Demand arrival and production time processes

We consider four different MAPs with high ($scv > 1$) and low ($scv < 1$) variability, and positive and negative first-lag autocorrelation in our analysis. We consider systems with correlated inter-arrivals and exponential processing times (MAP/M/1), Poisson arrivals and correlated processing times (M/MAP/1), and correlated arrival and processing times (MAP/MAP/1). This setup allows us to capture the impact of autocorrelation in demand arrivals or production times separately and jointly.

The MAP representation, the squared coefficient of variation scv, and the first-lag autocorrelation of the base MAPs, ρ_1 used in the analysis are given in Table 1. The traffic intensity of the system is set to be $\rho = 0.8$ in all cases. The mean of the inter-event times that have these MAPs is set to 1. We rescale the mean of inter-event times of the MAPs (Horvath and Telek, 2017) to generate systems with the given traffic intensity. This rescaling preserves coefficient of variation and autocorrelation structure of the original process. It is done by multiplying the elements of the D_0 and D_1 matrices with the ratio of the original mean and the new mean and normalizing the D_0 matrix to make $D_0 + D_1$ an infinitesimal generator matrix. Normalization is done by replacing the diagonal element of the new D_0 matrix by the negative of the summation of the other elements in a given row.

5.1.3. Generating MAPs with different first-lag autocorrelations

We analyze the impact of autocorrelation on the performance of the system by generating different MAPs with the same inter-event time distribution and different magnitudes of autocorrelation and analyzing the system with these generated MAPs following a similar approach used in Manafzadeh Dizbin and Tan (2019). The D_1 matrix of a MAP with the same distribution and zero autocorrelation denoted by D_1^{ren} is calculated as

$$D_1^{ren} = D_1 \mathbb{1} \beta, \tag{22}$$

where β is defined in Eq. (4). MAP(D_0, D_1^{ren}) is referred as MAP$^{\mathrm{ren}}$.

Table 1
The MAP representation, squared coefficient of variation, and first-lag autocorrelation of the MAPs employed in numerical analysis.

Process	D_0	D_1	$E[T]$	scv	ρ_1
Positively correlated MAP with $scv < 1$	$\begin{bmatrix} -1.4968 & 0 & 0.0426 \\ 0.0033 & -1.5339 & 1.4213 \\ 0 & 0 & -1.5340 \end{bmatrix}$	$\begin{bmatrix} 1.4213 & 0.0329 & 0 \\ 0 & 0 & 0.1093 \\ 0.0533 & 1.4807 & 0 \end{bmatrix}$	1	0.76	0.10
Positively correlated MAP with $scv > 1$	$\begin{bmatrix} -0.4531 & 0.0395 \\ 0 & -1.2612 \end{bmatrix}$	$\begin{bmatrix} 0.4135 & 0 \\ 0.0176 & 1.2436 \end{bmatrix}$	1	1.50	0.15
Negatively correlated MAP with $scv < 1$	$\begin{bmatrix} -1.5 & 1.5 & 0 \\ 0 & -3 & 1.5 \\ 0 & 0 & -1.5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \end{bmatrix}$	1	0.78	−0.14
Negatively correlated MAP with $scv > 1$	$\begin{bmatrix} -0.5214 & 0.5214 & 0 \\ 0 & -21.1159 & 0 \\ 0 & 0 & -21.1159 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 1.3035 & 0 & 19.8124 \\ 19.5518 & 0 & 1.5641 \end{bmatrix}$	1	2.75	−0.29

The first-lag autocorrelation of a MAP is a linear function of the elements of D_1 matrix (Horvath et al., 2005). When the first-lag autocorrelation of MAP(D_0, D_1) is ρ_1, in order to generate MAP(D_0, D_1^{new}) with the same inter-event time distribution but with the first-lag autocorrelation of $\theta\rho_1$ $(0 < \theta < 1)$, D_1^{new} matrix is constructed as

$$D_1^{new} = \theta D_1 + (1-\theta) D_1^{ren} \tag{23}$$

where D_1^{ren} is given in Eq. (22).

5.2. Impact of correlated demand arrival process

In this section, we investigate the impact of positive and negative autocorrelation in demand inter-arrival times on the performance of a MAP/M/1 system. In order to focus on the impact of autocorrelation in demand arrival process, the production times are set to be i.i.d. exponential random variables.

Tables 2, 3, 4, and 5 show the performance measures when a production system with positively or negatively correlated demand arrivals and exponential service times is controlled with the optimal policy and their comparisons with the results obtained with the benchmark policies for the cases when the squared coefficient of variation scv is less than 1 and greater than 1 respectively. The MAPs used for the demand arrival processes in these experiments are given in Table 1 and MAP$^{\text{ren}}$s are constructed by using the procedure given in Section 5.1.3.

Tables 2, 3, 4, and 5 show that the single-threshold policy that sets the single threshold based on the demand inter-arrival time distribution and also autocorrelation (Benchmark policy STWA) performs satisfactorily for all cases. The percentage error obtained for the total cost is 2%–3% for the positively correlated demand arrival cases and less than 0.1% for the negatively correlated demand arrival cases. However, when the autocorrelation is not incorporated in the production policy, as in benchmark policies MTNA and STNA, the percentage errors increase. These errors are more significant for the positively correlated demand arrival cases. The effect of ignoring autocorrelation on the expected inventory and backlog levels are more pronounced. For example, when the autocorrelation is not incorporated in production policy, the resulting backlog level is 101% higher than the optimal backlog level for the positively correlated demand arrival case with a high squared coefficient of variation (Table 3). Similarly, while a single-threshold policy that does not incorporate autocorrelation (STNA) yields 5% error for the total cost for the negatively correlated demand arrival case with high variability (Table 5), the resulting inventory level is 40% higher than the optimal inventory level.

The deviations between the results obtained by using the optimal policy and the benchmark policies are caused by the differences in the threshold levels set by the optimal and benchmark policies. For example, for the case with positively correlated demand arrivals with high variability (Table 3), the optimal policy uses two thresholds and sets them to (19, 12). A single-threshold policy incorporating the autocorrelation (STWA) sets the threshold to 16 that is in between the optimal thresholds. As a result, the percentage errors for the total cost, the expected backlog, and inventory levels are low (3%, 2%, and 5% respectively). However, for the positively correlated demand arrival case, when the autocorrelation is ignored in the production policy, using a state-dependent policy depending on the distribution of the demand inter-arrival time (MTNA) or using a single-threshold policy (STNA) do not yield good results since the thresholds are set far away from the optimal levels. For the case with negatively correlated demand arrivals (Tables 4 and 5), the threshold levels set by the benchmark policies are close to the optimal levels and therefore the percentage errors are smaller.

As shown in Fig. 3, in a system with positively correlated arrival process, the biggest optimal state-dependent threshold level set by the state-dependent threshold policy increases more than the increase of the optimal state-independent threshold level. This is due to the positive autocorrelation of demand arrivals that causes a short (long) inter-arrival time is to be followed by a short (long) inter-arrival time. Hence, the state-dependent threshold policy increases the biggest optimal state-dependent threshold level, greater than that of the optimal state-independent threshold level to cope with the variability in the inter-arrival times. For the negatively correlated demand arrival case, demand accumulation due to variability is reduced by the arrival pattern and as a result the errors are lower, and the single-threshold policy performs well.

Figs. 3 and 4 show the impact of incremental increase in the first-lag autocorrelation of positively and negatively correlated demand arrival processes respectively. In order to conduct these experiments, the MAPs given in Table 1 are used and the new MAPs are generated by using the procedure given in Section 5.1.3. The distance between the zero and the first-lag autocorrelation of MAP is divided into 9 equal distances to generate MAPs with incrementally increasing the first-lag autocorrelations. The figures show the effects of autocorrelation on the threshold levels and percentage difference between the optimal total cost of the system, TC^* controlled with the optimal state-dependent policy and the cost of the state-independent threshold policy, TC^*_S, where the optimal threshold is set by incorporating the demand inter-arrival distribution and autocorrelation (STWA). The performance of STWA deteriorates in general as the positive correlation of the system increases. However, the policy does not show a monotone behavior for negatively correlated systems. This erratic behavior maybe the result of the significant decrease of the cost for negative correlated systems.

5.3. Impact of correlated production time process

We now focus on the impact of autocorrelation in production time process on the performance of a M/MAP/1 system controlled with the optimal policy and the benchmark policies. Similar to the analysis of the effect of correlated demand arrivals, in order to focus on the impact of autocorrelation in the service process, the demand inter-arrival times are set to be i.i.d. exponential random variables.

Tables 6, 7, 8, and 9 show the performance measures when a production system with positively or negatively correlated production

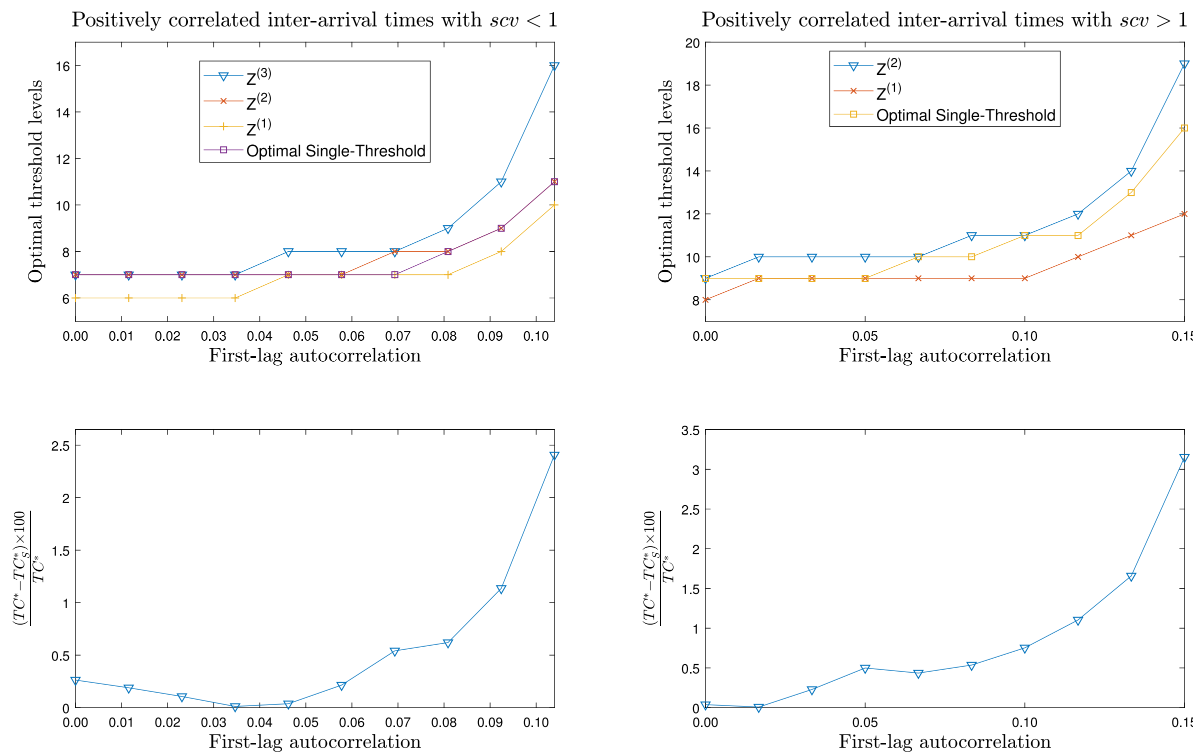

Fig. 3. Impact of the first-lag autocorrelation of a positively correlated demand inter-arrival process on the threshold levels and the percentage difference between the total costs of MAP/M/1 system controlled with the optimal state-dependent threshold policy and the single-threshold with autocorrelation (STWA) policy ($h = 1$, $b = 5$).

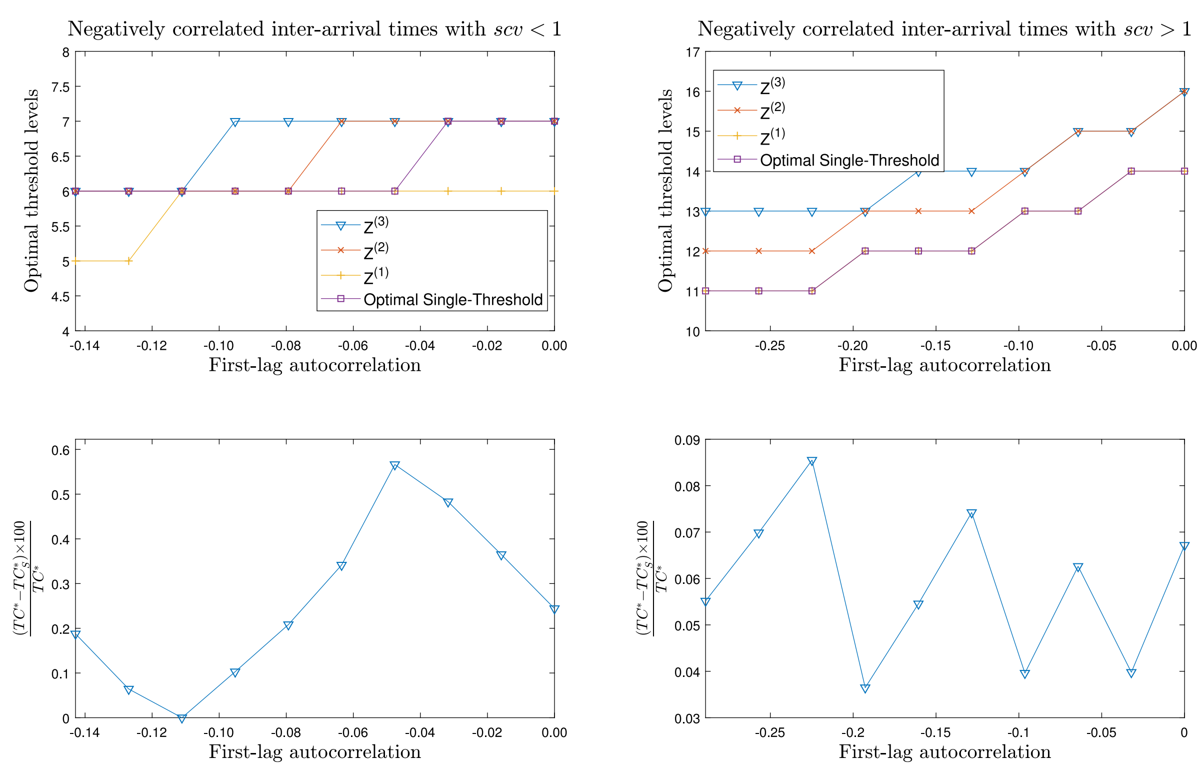

Fig. 4. Impact of the first-lag autocorrelation of a negatively correlated demand inter-arrival process on the threshold levels and the percentage difference between the total costs of MAP/M/1 system controlled with the optimal state-dependent threshold policy and the single-threshold with autocorrelation (STWA) policy ($h = 1$, $b = 5$).

Table 2
Performance measures of a positively correlated demand inter-arrival and exponential production time system (MAP/M/1) controlled with the state-dependent threshold policy and benchmark policies (Demand inter-arrival time $scv < 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	MAP/M/1	(16, 11, 10)	13.4067		1.3201		6.8064		0.1490	
MTNA	MAP^{ren}/M/1	(7, 7, 6)	14.9538	12%	2.2120	68%	3.8941	−43%	0.2511	69%
STWA	MAP/M/1	11	13.7295	2%	1.3981	6%	6.7392	−1%	0.1577	6%
STNA	MAP^{ren}/M/1	7	14.9017	11%	2.2601	71%	3.6013	−47%	0.2566	72%

Table 3
Performance measures of a positively correlated demand inter-arrival and exponential production time system (MAP/M/1) controlled with the state-dependent threshold policy and benchmark policies (Demand inter-arrival time $scv > 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	MAP/M/1	(19, 12)	17.8130		1.7380		9.1231		0.1583	
MTNA	MAP^{ren}/M/1	(9, 8)	21.4375	20%	3.4962	101%	3.9565	−57%	0.3184	101%
STWA	MAP/M/1	16	18.3744	3%	1.7677	2%	9.5361	5%	0.1610	2%
STNA	MAP^{ren}/M/1	9	21.4637	20%	3.4492	98%	4.2176	−54%	0.3142	98%

Table 4
Performance measures of a negatively correlated demand inter-arrival and exponential production time system (MAP/M/1) controlled with the state-dependent threshold policy and benchmark policies (Demand inter-arrival time $scv < 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	MAP/M/1	(6, 6, 5)	6.1660		0.6114		3.1088		0.1543	
MTNA	MAP^{ren}/M/1	(7, 7, 6)	6.2399	1%	0.4571	−25%	3.9544	27%	0.1154	−25%
STWA	MAP/M/1	6	6.1775	0%	0.5692	−7%	3.3316	7%	0.1437	−7%
STNA	MAP^{ren}/M/1	7	6.3155	2%	0.4255	−30%	4.1879	35%	0.1074	−30%

Table 5
Performance measures of a negatively correlated demand inter-arrival and exponential production time system (MAP/M/1) controlled with the state-dependent threshold policy and benchmark policies (Demand inter-arrival time $scv > 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	MAP/M/1	(13, 12, 11)	11.6780		1.0670		6.3431		0.1513	
MTNA	MAP^{ren}/M/1	(13, 16, 16)	12.3225	6%	0.6743	−37%	8.9509	41%	0.0956	−37%
STWA	MAP/M/1	11	11.6845	0%	1.0762	1%	6.3034	−1%	0.1526	1%
STNA	MAP^{ren}/M/1	14	12.3084	5%	0.6802	−36%	8.9074	40%	0.0965	−36%

times and exponential demand inter-arrival times is controlled with the optimal and the benchmark policies for the cases $scv < 1$ and $scv > 1$ respectively. The MAPs used for the production time processes are given in Table 1 and MAP^{ren}s are constructed by using the procedure given in Section 5.1.3.

Similar to the effect of autocorrelation in the demand arrivals, the effect of ignoring positive autocorrelation in production time process in the production policy is more significant compared to the effect of negative autocorrelation. The single-threshold policy where the threshold is set based on the service time distribution and autocorrelation (STWA) yields a total cost that is 2%–4% higher than the optimal cost. The resulting expected backlog and inventory levels are also within 6% of the optimal levels. However, if the autocorrelation information is not used in selecting the policy parameters, as in the benchmark policies MTNA and STNA, the threshold levels set by these policies are far away than the optimal levels. Accordingly, the total cost obtained by using these policies is 9%–21% above the optimal cost for the positively correlated service times.

When the service times are negatively correlated, the thresholds set by the benchmark policies are closer to the optimal levels and the total costs are within 6% of the optimal cost. For positively and negatively correlated service times, ignoring autocorrelation in production policies yield significant errors for the expected backlog and inventory levels.

Figs. 5 and 6 show the impact of incremental increase in the first-lag autocorrelation of positively and negatively correlated service processes on the threshold levels and on the percentage difference between the total costs of systems controlled by the optimal state-dependent and state-independent threshold policy with autocorrelation (STWA). Fig. 5 shows that the optimal threshold levels of the system increase as the first-lag autocorrelation increases. For the positively correlated service times, the performance of the single-threshold policy with autocorrelation (STWA) in controlling the system deteriorates as the first-lag autocorrelation increases as well. For the negatively correlated service times, Fig. 6 shows that the optimal state-dependent threshold-levels also increases as a function of the first-lag autocorrelation and the performance of the single-threshold policy in controlling the system deteriorates as the first-lag autocorrelation increases. However, the relation is not monotone.

5.4. Impact of correlated arrival and service processes on the optimal control of the system

Finally, in this section, we focus on the combined effects of autocorrelation in both the demand arrival and also production processes on the performance of a MAP/MAP/1 production system controlled with the optimal and the benchmark policies. We consider four different cases: positively correlated arrival and service, positively correlated arrival and negatively correlated service, negatively correlated arrival and positively correlated service, and negatively correlated arrival and service. The MAPS given in Table 1 are used for both demand arrival and service time processes in these experiments.

Fig. 7 shows the percentage increase in the total cost of a correlated system controlled by the threshold levels that are optimal for the uncorrelated system (Benchmark Policy MTNA) for the low variability case $scv < 1$. Since the MAPs given in Table 1 for $scv < 1$ have 3 states, the optimal state-dependent threshold policy that incorporates

Table 6
Performance measures of a positively correlated production time and exponential demand inter-arrival time system (M/MAP/1) controlled with the state-dependent threshold policy and benchmark policies (Production time $scv < 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	M/MAP/1	(12, 11, 7)	10.1854		1.0483		4.9438		0.1593	
MTNA	M/MAP^{ren}/1	(8, 7, 6)	11.3075	11%	1.6547	58%	3.0339	−39%	0.2516	58%
STWA	M/MAP/1	9	10.5654	4%	1.0761	3%	5.1849	5%	0.1630	2%
STNA	M/MAP^{ren}/1	7	11.0796	9%	1.4951	43%	3.6039	−27%	0.2268	42%

Table 7
Performance measures of a positively correlated production time and exponential demand inter-arrival time system (M/MAP/1) controlled with the state-dependent threshold policy and benchmark policies (Production time $scv > 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	M/MAP/1	(30, 21)	29.4551		3.1322		13.7943		0.1626	
MTNA	M/MAP^{ren}/1	(12, 9)	34.3890	17%	5.9574	90%	4.6020	−67%	0.3125	92%
STWA	M/MAP/1	22	29.9031	2%	3.1155	−1%	14.3256	4%	0.1617	−1%
STNA	M/MAP^{ren}/1	9	35.6392	21%	6.2382	99%	4.4482	−68%	0.3269	101%

Table 8
Performance measures of a negatively correlated production time and exponential demand inter-arrival time system (M/MAP/1) controlled with the state-dependent threshold policy and benchmark policies (Production time $scv < 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	M/MAP/1	(7, 6, 5)	6.2734		0.6233		3.1572		0.1537	
MTNA	M/MAP^{ren}/1	(7, 7, 6)	6.2925	0%	0.5525	−11%	3.5302	12%	0.1362	−11%
STWA	M/MAP/1	6	6.3932	2%	0.6230	0%	3.2780	4%	0.1536	0%
STNA	M/MAP^{ren}/1	7	6.4719	3%	0.4695	−25%	4.1245	31%	0.1157	−25%

Table 9
Performance measures of a negatively correlated production time and exponential demand inter-arrival time system (M/MAP/1) controlled with the state-dependent threshold policy and benchmark policies (Production time $scv > 1$).

Policy	System	$\bar{Z}^*$	$TC(\bar{Z}^*)$	Δ_{TC}	$X^-(\bar{Z}^*)$	Δ_{X^-}	$X^+(\bar{Z}^*)$	Δ_{X^+}	$P_0(\bar{Z})$	Δ_{P_0}
Optimal	MAP/MAP/1	(12, 11, 9)	11.1589		1.0147		6.0855		0.1461	
MTNA	MAP^{ren}/MAP/1	(13, 12, 12)	11.6535	4%	0.7389	−27%	7.9593	31%	0.1064	−27%
STWA	MAP/MAP/1	11	11.4695	3%	1.0019	−1%	6.4602	6%	0.1442	−1%
STNA	MAP^{ren}/MAP/1	13	11.8632	6%	0.7341	−28%	8.1925	35%	0.1057	−28%

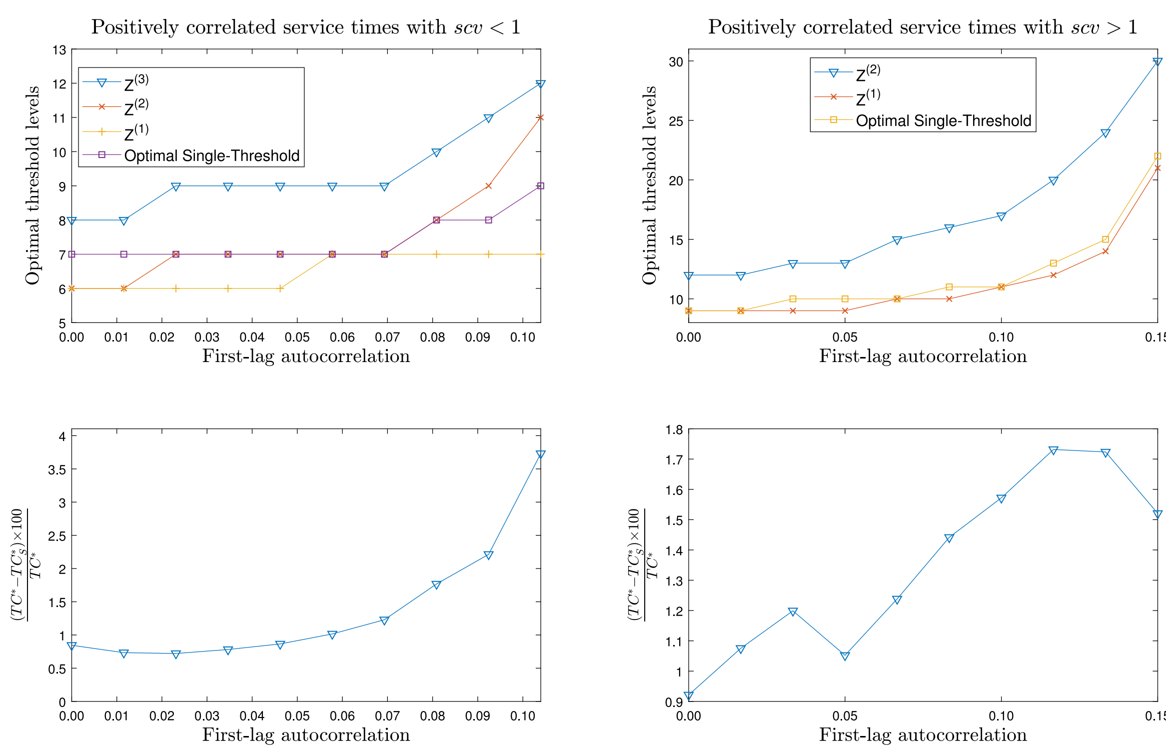

Fig. 5. Impact of the first-lag autocorrelation of a positively correlated production time process on the threshold levels and the percentage difference between the total costs of M/MAP/1 system controlled with the optimal state-dependent threshold policy and the single-threshold policy with autocorrelation (STWA) policy ($h = 1$, $b = 5$).

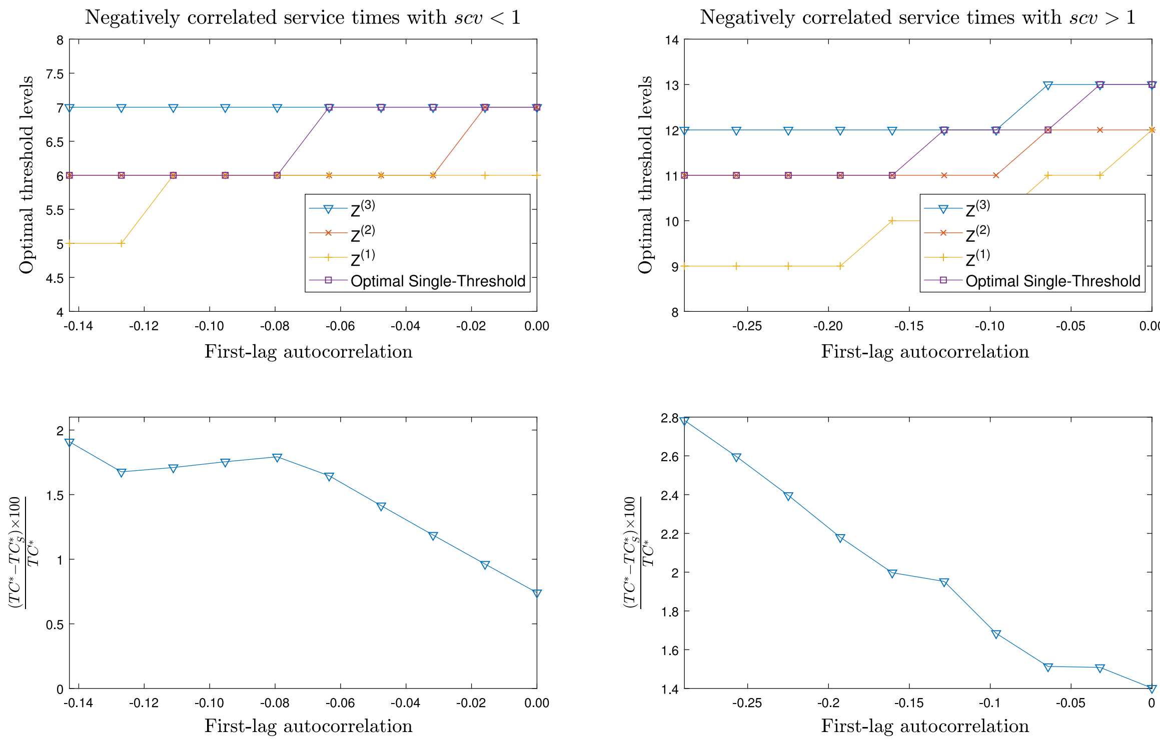

Fig. 6. Impact of the first-lag autocorrelation of a negatively correlated production time process on the threshold levels and the percentage difference between the total costs of M/MAP/1 system controlled with the optimal state-dependent threshold policy and the state-independent threshold policy with autocorrelation (STWA) policy ($h = 1$, $b = 5$).

autocorrelation in service and demand arrival times sets 9 thresholds depending on the state of the demand and the production time process. The figure shows that positive correlation has more impact on the performance of the system. Controlling a system with positively correlated arrival and service times with first-lag autocorrelation of 0.1 results in a 20% increase in the total cost. Results are less dramatic for a system with negatively correlated arrival and service times. Controlling a system with negatively correlated arrival and service times with the first-lag autocorrelation of −0.14 result in a 6% increase in the total cost of the system. In addition to the impact of ignoring correlation, Fig. 7 captures the interaction between negative and positive correlations as well. For the cases where the signs of the demand arrival and service process autocorrelations are different, i.e., positive/negative and negative/positive, positive and negative correlations neutralize the impact of each other, resulting in a lesser increase in the deviations. Similar results follow for processes with $scv > 1$.

Fig. 8 demonstrates the percentage difference between the total cost of MAP/MAP/1 system with correlated inter-arrival and service times controlled with the optimal policy and the single threshold policy (STWA). Similar to the previous cases the performance of the single threshold policy deteriorates as the correlation increase. Controlling a system where both of the arrival and service processes are positively correlated with a first-lag of 0.1 with a single threshold policy results in an 8% increase in the cost in comparison to the optimal policy. Fig. 9 demonstrates the percentage difference between the total cost of the correlated system with that of the system controlled by single threshold policy that assumes i.i.d. inter-event times (STNA). The policy performs well for cases with low magnitude of correlation and cases that negative and positive correlations interact with each other. Its performance gets worse as the correlation increases.

6. Conclusions

In this paper, we investigate the optimal control policy of a system with correlated demand inter-arrival and processing times modeled as Markovian Arrival Processes. We prove that the optimal policy that minimizes the expected average cost of the production-inventory system in the long run is a state-dependent threshold policy. We then propose a matrix analytic method to evaluate the performance of the system controlled by the state-dependent threshold policy and use the policy iteration algorithm to determine the optimal thresholds.

We use the structural properties of MAP to generate demand inter-arrival and production time processes with the same distribution and different magnitudes of autocorrelation. These MAPs are then used to evaluate the impact of autocorrelation in the inter-arrival and processing times on the optimal threshold levels and performance of the state-independent policy.

Our numerical analysis demonstrates that the optimal performance measures of a system with correlated inter-arrival times and service process are dependent on the autocorrelation structure in the system. Positive autocorrelations in the inter-arrival and processing times increase the optimal threshold levels of the system. In contrast, negative autocorrelations in inter-arrival and processing times decrease the optimal threshold levels. In this study, we report the results for a system with moderate traffic intensity. Our extensive numerical results showed that the impact of the correlation in this system increases significantly as we increase the traffic intensity of the system.

We evaluate the performance of the optimal single-threshold policy in controlling the system by comparing the total costs of systems controlled by 3 benchmark policies: a state-dependent policy that uses the distribution but assumes i.i.d. inter-event times, a single-threshold policy that is set by using both the distribution and also the autocorrelation, and a single-threshold policy that uses the distribution but assume that the inter-event times are i.i.d.

Our analysis demonstrates that ignoring autocorrelation in setting the parameters of the production policy causes significant errors in the expected inventory and backlog costs. A single-threshold policy that sets the threshold based on the distribution and also the autocorrelation performs satisfactorily. However, ignoring positive correlation in setting the state-dependent thresholds levels and the single threshold

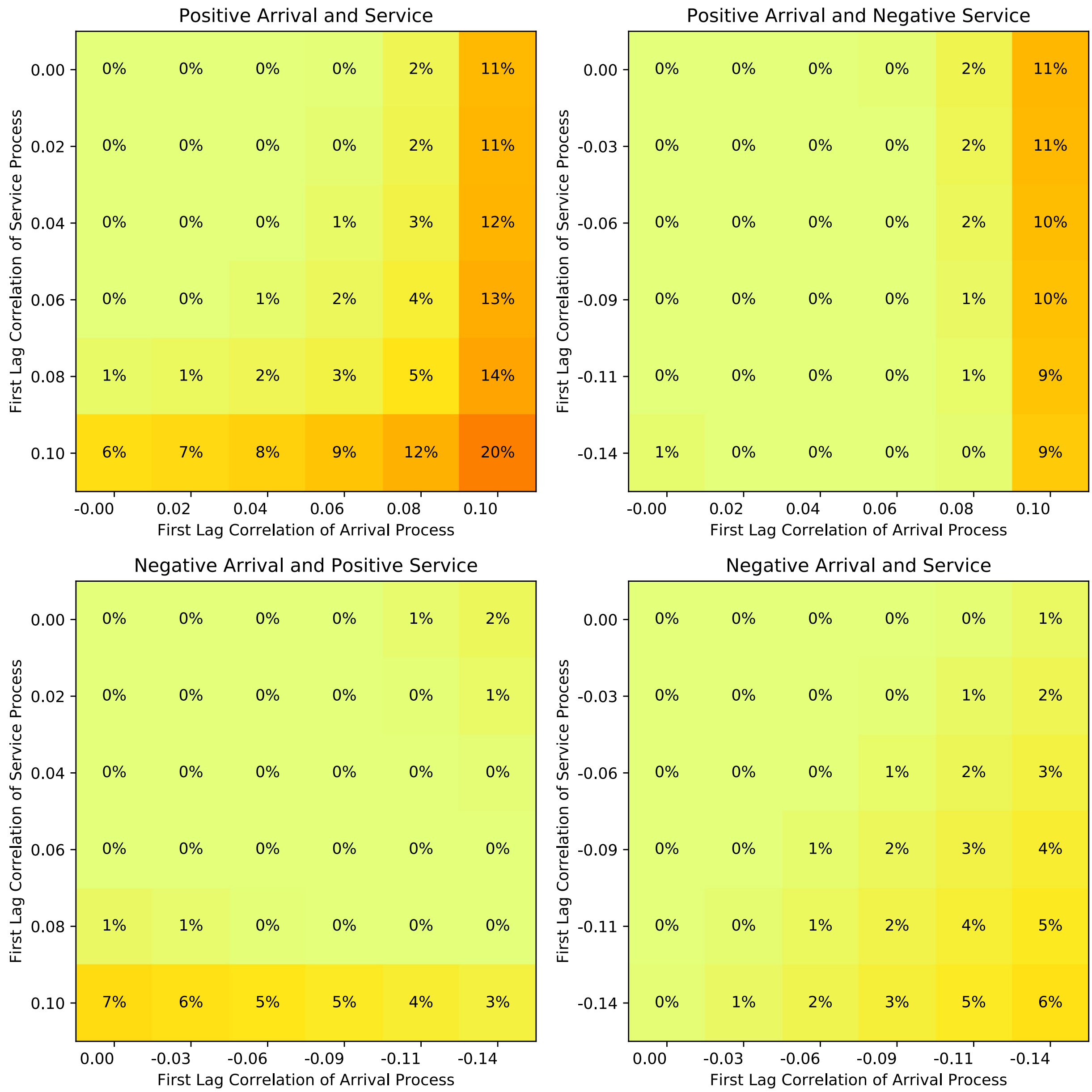

Fig. 7. Percentage difference between the total cost of MAP/MAP/1 system with correlated inter-arrival and service times controlled with the optimal policy and the Multiple-Threshold NO Autocorrelation policy (MTNA).

based on the distribution yields high errors. The difference between the total costs of the systems controlled by the state-dependent and the state-independent policies may increase as the magnitude of the first-lag autocorrelation increases. Our study shows that an effective production control policy must take correlations in service and demand processes into account.

This research can be extended in a number of directions. The approach used in deriving the optimal policy can be extended to determine the optimal threshold levels in a partially observable system, where the buffer level is observable while the state of the underlying Markovian process is unobservable. The methodology used here for evaluating the optimal policy can also be adopted to evaluate the optimal control policy of a system with different classes of customers. In order to implement the state-dependent threshold policy or an approximate policy that uses the autocorrelation and distribution information, a methodology needs to be developed to determine the thresholds by using the collected data from the shop floor. These are left for future research.

CRediT authorship contribution statement

Nima Manafzadeh Dizbin: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - review & editing. **Barış Tan:** Conceptualization, Methodology, Writing - review & editing, Supervision, Funding acquisition.

Acknowledgments

Research leading to these results has received funding from the EU ECSEL Joint Undertaking under grant agreement no. 737459 (project Productive4.0) and from TUBITAK, Turkey (217M145).

Appendix. Proof of Theorem 1

We adopt the methodology presented in Koole (2006) to identify the optimal control policy of a system with correlated demand-arrival and service processes modeled as Markovian Arrival Process (MAP). Recall

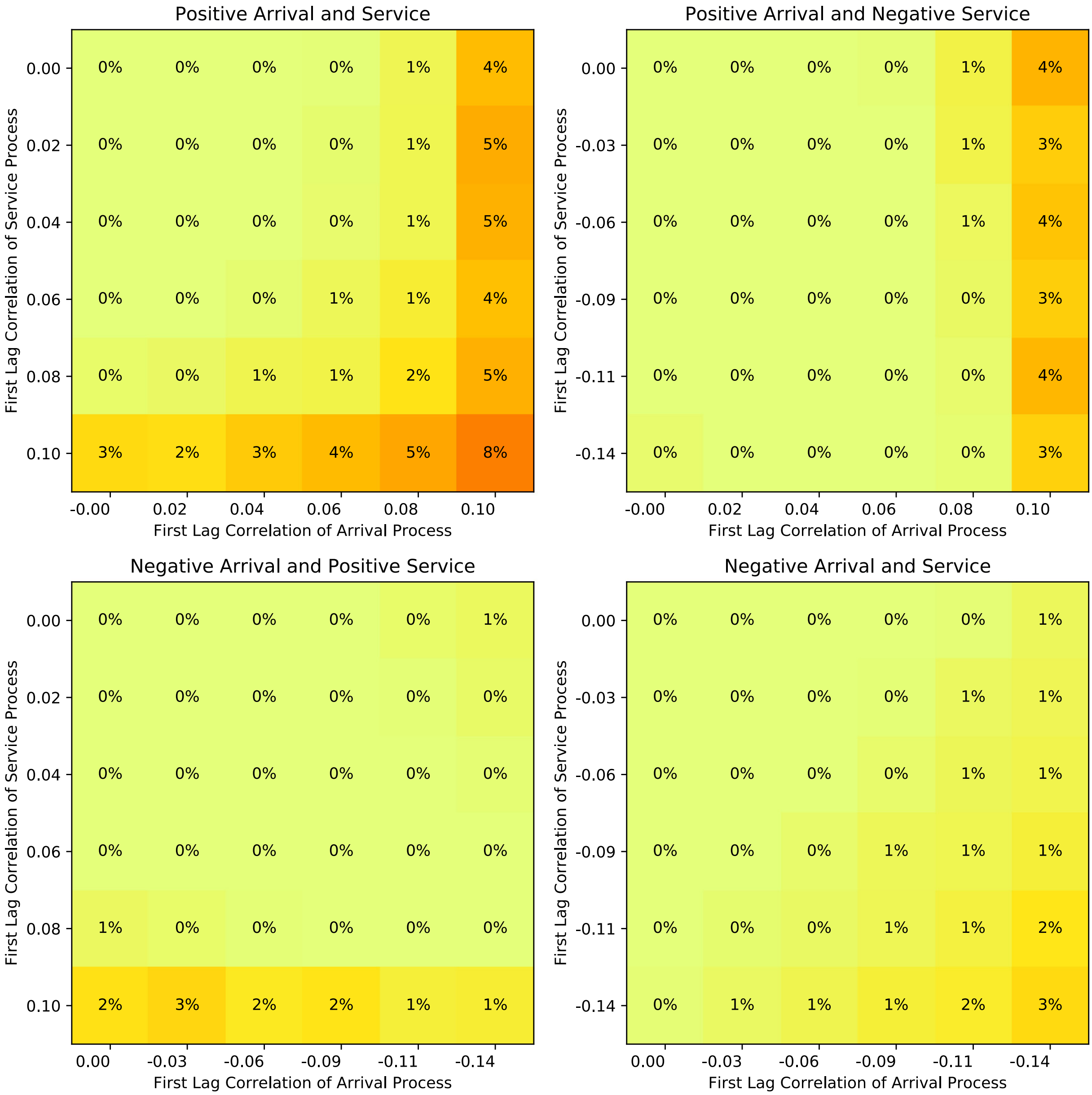

Fig. 8. Percentage difference between the total cost of MAP/MAP/1 system with correlated inter-arrival and service times controlled with the optimal policy and the Single-Threshold With Autocorrelation policy (STWA).

that the state of the production-inventory system presented in Section 3 can be fully specified by the inventory level (X) at the buffer and state of the underlying Markovian process corresponding to demand-arrival (j_a) and service (j_s) processes. We represent the state of the system with (X, j_a, j_s).

$$V(X, j_a) + g$$
$$= \min \begin{cases} \frac{C(X)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j) \\ \quad + \sum_{j \in J_a} P_1(j_a, j) V(X-1, j) & u = 0, \\ \frac{C(X)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j) \\ \quad + \sum_{j \in J_a} P_1(j_a, j) V(X-1, j) \\ \quad + (\frac{\mu}{\alpha}) V(X+1, j_a) - (\frac{\mu}{\alpha}) V(X, j_a) & u = 1. \end{cases} \tag{A.1}$$

For simplicity, we first analyze a system with $MAP(D_0, D_1)$ arrival and exponential service process with rate μ. The state of the system in this case can be represented by the inventory level at the buffer and the state of the arrival process (X, j_a). The action space of the system consists of two actions denoted by $u = 0$, and $u = 1$. The action $u = 0$ corresponds to stopping or not continuing the production, and the action $u = 1$ corresponds to starting or continuing the production. The optimality equation of this system can be written as in Eq. (A.1) using the uniformization technique. Note that the uniformization operator preserves the monotonicity and convexity characteristics of the value function of the continuous-time model (Koole, 2006). Hence, the optimality results carry over to the continuous time problem. Starting from state (X, j_a) the system spends an exponential time with rate α in this state, which results in $\frac{C(X)}{\alpha}$ cost, before moving to the next state. The transition probabilities to the next state is determined by the action u. If $u = 0$ the transition probabilities of the system are related to the demand-arrival. In this case, the transition probabilities corresponding to state-change without new demand-arrivals and with demand-arrivals are captured with non-diagonal elements of the P_0 matrix, and P_1 matrix, respectively. The system stays at the same state with probability $1 + \frac{D_0(j_a, j_a)}{\alpha} = P_0(j_a, j_a)$. The transition probabilities

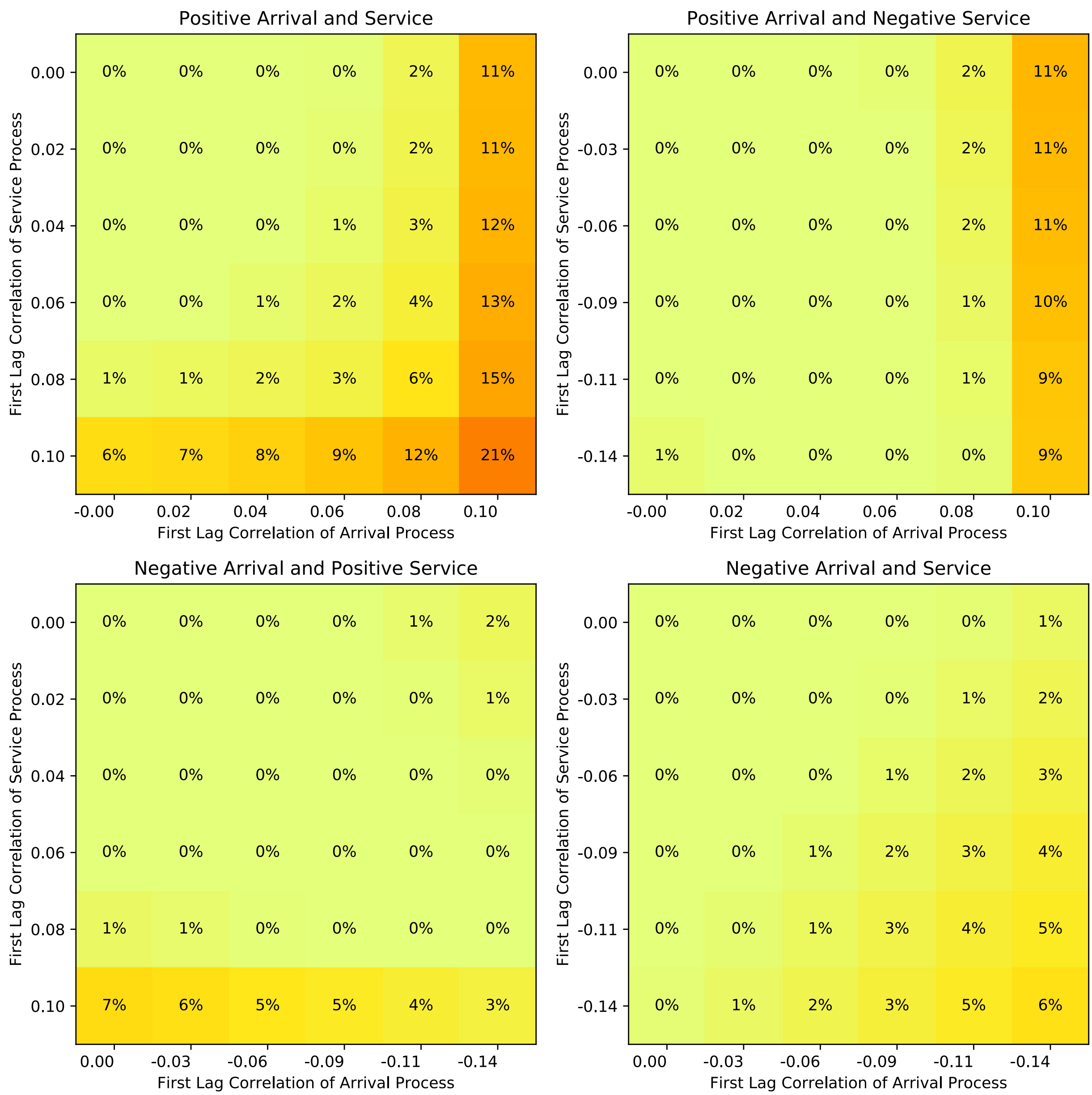

Fig. 9. Percentage difference between the total cost of MAP/MAP/1 system with correlated inter-arrival and service times controlled with the optimal policy and the Single-Threshold No Autocorrelation policy (STNA).

related to demand-arrival are the same when action $u = 1$ is taken. In this case, the system moves to state $(X+1, j_a)$ with probability $\frac{\mu}{\alpha}$ and remain in the same state with probability $1 - \frac{\mu - D_0(j_a, j_a)}{\alpha} = P_0(j_a, j_a) - \frac{\mu}{\alpha}$.

By rewriting Eq. (A.1) and letting $V_{n+1}(X, i)$ be the minimal expected total cost if there are $n+1$ more events to go, $V_{n+1}(X, j_a)$ can be written in terms of $V_n(X, j_a)$ as in Eq. (A.2).

$$V_{n+1}(X, j_a) + g = \frac{C(X)}{\alpha} + \sum_{j \in J_a} P_0(i, j) V_n(X, j) + \sum_{j \in J_a} P_1(j_a, j) V_n(X-1, j)$$
$$+ \frac{\mu}{\alpha} V_n(X, j_a) + (\frac{\mu}{\alpha}) \min\{V_n(X, j_a), V_n(X+1, j_a)\}. \tag{A.2}$$

Eq. (A.2) can be expressed by using the operators given in Eq. (A.3) which are similar to operators defined in Koole (2006). These operators preserve the convexity and increasing property of a given function f. Let x be the inventory level and j represent the state of the Markovian process.

$$\begin{aligned}
T_{costs} f(x) &= C(x) + f(x), \\
T_{unif}(f_1, \dots, f_l)(x) &= \sum_{j=1}^{l} p(j) f_j(x), \quad where \quad \sum_{j=1}^{l} p(j) = 1, \\
T_{env} f(x, j) &= f(x, j'), \\
T_A f(x, j) &= f(x-1, j), \\
T_P f(x, j) &= f(x+1, j), \\
T_{CP} f(x, j) &= \min\{f(x, j), T_P(T_{env})\} = \min\{f(x, j), f(x+1, j')\}, \\
T_{menv} f(x, j) &= \min\{f(x, j), T_{env}\} = \min\{f(x, j), f(x, j')\}, \\
T_{DA} f(x, j) &= T_A(T_{env}) = f(x-1, j').
\end{aligned} \tag{A.3}$$

Eq. (A.2) can be written in terms of the operator in Eq. (A.3) as follows:

$$V_{n+1} + g = T_{costs}\Big(T_{unif}\left(T_{env}V_n, \dots, T_{env}V_n, T_{DA}V_n, \dots, T_{DA}V_n\right)$$
$$+ \frac{\mu}{\alpha}\left(V_n + T_{CP}V_n\right)\Big). \tag{A.4}$$

We show by induction that $V_{n+1}(X, j_a)$ is convex in X for a given $j_a \in J_a$. Let $V_0(X, j_a) = C(X)$. Convexity of the $V_1(X, j_a)$ is established trivially. Assume by induction that $V_{n'}(X, j_a)$ is convex in X for $n' \in$

$(2 \dots n)$ and $j_a \in J_a$. Since all of the operators preserve the convexity property, convexity of V_{n+1} is followed. For $n \to \infty$ this policy converges to the policy that minimizes the long-run average cost of the system. By Theorem 8.1 in Koole (2006), convexity of $V(X, j_a)$ in X results in a threshold type optimal policy.

The optimality equation for a system with $MAP(A_0, A_1)$ service process differs from the optimality equation of a system with exponential service process only in the second line of Eq. (A.1). The state of the system in this case is (X, j_a, j_s). The second line of the optimality equation of this system includes transition probabilities related to state-change (off-diagonal elements of R_0) and service-completion (R_1) of the service process and probability of the state remaining the same $(1 - \frac{-D_0(j_a,j_a)-A_0(j_s,j_s)}{\alpha})$.

$$\begin{aligned}
V(X, j_a, j_s) + g &= \frac{C(x)}{\alpha} + \sum_{j \in J_a/\{j_a\}} P_0(j_a, j) V(X, j, j_s) \\
&+ \sum_{j \in J_a} P_1(j_a, j) V(X-1, j, j_s) \\
&+ \sum_{j' \in J_s/\{j_s\}} R_0(j_s, j') V(X, j_a, j') + \sum_{j' \in J_s} R_1(j_s, j') V(X+1, j_a, j') \\
&+ (1 - \frac{-D_0(j_a, j_a) - A_0(j_s, j_s)}{\alpha}) V(X, j_a, j_s),
\end{aligned} \tag{A.5}$$

where J_s is the set of Markovian states of the service process. Hence, the optimality equation is of the following form:

$$\begin{aligned}
V(X, j_a, j_s) + g &= \frac{C(x)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j, j_s) \\
&+ \sum_{j \in J_a} P_1(j_a, j) V(X-1, j, j_s) + \\
&\min \Bigg\{ 0, \sum_{j' \in J_s/\{j_s\}} R_0(j_s, j') V(X, j_a, j') + \sum_{j' \in J_s} R_1(j_s, j') V(X+1, j_a, j') \\
&- (\frac{-A_0(j_a, j_a)}{\alpha}) V(X, j_a, j_s) \Bigg\}
\end{aligned} \tag{A.6}$$

By using the following equation

$$\sum_{j' \in J_s/\{j_s\}} R_0(j_s, j') + \sum_{j' \in J_s} R_1(j_s, j') = \frac{-A_0(j_s, j_s)}{\alpha}, \tag{A.7}$$

we can re-write Eq. (A.6) as

$$\begin{aligned}
V(X, j_a, j_s) + g &= \frac{C(x)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j, j_s) \\
&+ \sum_{j \in J_a} P_1(j_a, j) V(X-1, j, j_s) + \\
&\min \Bigg\{ 0, \sum_{j' \in J_s/\{j_s\}} R_0(j_s, j') \big(V(X, j_a, j') - V(X, j_a, j_s) \big) \\
&+ \sum_{j' \in J_s} R_1(j_s, j') \big(V(X+1, j_a, j') - V(X, j_a, j_s) \big) \Bigg\}
\end{aligned} \tag{A.8}$$

By taking the minimum function inside each function and adding and subtracting $(\frac{A_0(j_s,j_s)}{\alpha}) V(X, j_a, j_s)$ we get the following equation.

$$\begin{aligned}
V(X, j_a, j_s) + g &= \frac{C(x)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j, j_s) \\
&+ \sum_{j \in J_a} P_1(j_a, j) V(X-1, j, j_s) - (\frac{A_0(j_s, j_s)}{\alpha}) V(X, j_a, j_s) \\
&+ \sum_{j' \in J_s/\{j_s\}} R_0(j_s, j') \min \Big\{ V(X, j_a, j_s), V(X, j_a, j') \Big\} \\
&+ \sum_{j' \in J_s} R_1(j_s, j') \min \Big\{ V(X, j_a, j_s), V(X+1, j_a, j') \Big\}
\end{aligned} \tag{A.9}$$

which can be written in the following form:

$$\begin{aligned}
V(X, j_a, j_s) + g &= \frac{C(x)}{\alpha} + \sum_{j \in J_a} P_0(j_a, j) V(X, j, j_s) \\
&+ \sum_{j \in J_a} P_1(j_a, j) V(X-1, j, j_s) \\
&+ \sum_{j' \in J_s} R_0(j_s, j') \min \Big\{ V(X, j_a, j_s), V(X, j_a, j') \Big\} \\
&+ \sum_{j' \in J_s} R_1(j_s, j') \min \Big\{ V(X, j_a, j_s), V(X+1, j_a, j') \Big\}
\end{aligned} \tag{A.10}$$

Eq. (A.10) can be written as function of the given operators in Eq. (A.3) as given in Eq. (A.11). Convexity of the V_{n+1} follows from the fact that the operators preserve convexity.

$$\begin{aligned}
V_{n+1} + g &= T_{costs} \big(T_{unif} \left(T_{env} V_n, \dots, T_{env} V_n, T_{DA} V_n, \dots, T_{DA} V_n \right) \\
&+ T_{unif} \left(T_{menv} V_n, \dots, T_{menv} V_n, T_{CP} V_n, \dots, T_{CP} V_n \right) \big).
\end{aligned} \tag{A.11}$$

By Theorem 8.1 of Koole (2006) the optimal policy is of the threshold type for a given Markovian state, proving Theorem 1.

References

Asmussen, S., Koole, G., 1993. Marked point processes as limits of Markovian arrival streams. J. Appl. Probab. 30, 365–372.

Baumann, H., Sandmann, W., 2010. Numerical solution of level dependent quasi-birth-and-death processes. In: Procedia Computer Science, Vol. 1. Elsevier B.V., pp. 1561–1569.

Bayraktar, E., Ludkovski, M., 2010. Inventory management with partially observed nonstationary demand. Ann. Oper. Res. 176, 7–39.

Berman, O., Kim, E., 2001. Dynamic order replenishment policy in internet-based supply chains. Math. Methods Oper. Res. (ZOR) 53, 371–390.

Bertsekas, D.P., 2005. Dynamic Programming and Optimal Control. Athena Scientific.

Beyer, D., Sethi, S.P., 1997. Average cost optimality in inventory models with markovian demands. J. Optim. Theory Appl. 92, 497–526.

Beyer, D., Sethi, S.P., Taksar, M., 1998. Inventory models with markovian demands and cost functions of polynomial growth. J. Optim. Theory Appl. 98, 281–323.

Bright, L., Taylor, P., 1995. Calculating the equilibrium distribution in level dependent quasi-birth-and-death processes. communications in statistics. Stoch. Models 11, 497–525.

Buchholz, P., Kriege, J., Felko, I., 2014. Input Modeling with Phase-Type Distributions and Markov Models: Theory and Applications. Springer Briefs in Mathematics.

Chen, F., Song, J.S., 2001. Optimal policies for multi-echelon inventory problems with Markov modulated demand. Oper. Res. 49, 226–234.

Cheng, F., Sethi, S., 1999. Optimality of state-dependent (s,S) policies in inventory models with Markov-modulated demand and lost sales. Prod. Oper. Manage. 8, 183–192.

Duri, C., Frein, Y., Di Mascolo, M., 2000. Comparison among three pull control policies: Kanban, base stock, and generalized kanban. Ann. Oper. Res. 93, 41–69.

Gershwin, S.B., 2000. Design and operation of manufacturing systems: The control-point policy. IIE Trans. 32, 891–906.

Gershwin, S.B., Tan, B., Veatch, M.H., 2009. Production control with backlog-dependent demand. IIE Trans. 41, 511–523.

Gurkan, G., Karaesmen, F., Ozdemir, O., 2007. Optimal threshold levels in stochastic fluid models via simulation-based optimization. Discrete Event Dyn. Syst. 17, 53–97.

He, Q.-M., Jewkes, E., 2000. Performance measures of a make-to-order inventory-production system. IIE Trans. 32, 409–419.

He, Q.-M., Jewkes, E., Buzacott, J., 2002. Optimal and near-optimal inventory control policies for a make-to-order inventory–production system. European J. Oper. Res. 141, 113–132.

Heindl, A., Zhang, Q., Smirni, E., 2004. ETAQA truncation models for the MAP/MAP/1 departure process. In: First International Conference on the Quantitative Evaluation of Systems, 2004. QEST 2004. Proceedings. IEEE, pp. 100–109.

Hendricks, K.B., McClain, J.O., 1993. The output process of serial production lines of general machines with finite buffers. Manage. Sci. 39, 1194–1201.

Horvath, G., Buchholz, P., Telek, M., 2005. A MAP fitting approach with independent approximation of the inter-arrival time distribution and the lag correlation. In: Second International Conference on the Quantitative Evaluation of Systems (QEST'05). pp. 124–133.

Horvath, G., Telek, M., 2017. BuTools 2: A rich toolbox for Markovian performance evaluation. In: Proceedings of the 10th EAI International Conference on Performance Evaluation Methodologies and Tools on 10th EAI International Conference on Performance Evaluation Methodologies and Tools VALUETOOLS'16. ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering, ICST, Brussels, Belgium, pp. 137–142.

Hu, J., Zhang, C., Zhu, C., 2016. (S, S) inventory systems with correlated demands. INFORMS J. Comput. 28, 603–611.

Inman, R.R., 1999. Empirical evaluation of exponential and independence assumptions in queueing models of manufacturing systems. Prod. Oper. Manage. 8, 409–432.

Janakiraman, G., Muckstadt, J., 2009. A decomposition approach for a class of capacitated serial systems. Oper. Res. 57, 1384–1393.
Jiang, Q., Xing, W., Hou, R., Zhou, B., 2015. An optimization model for inventory system and the algorithm for the optimal inventory costs based on supply-demand balance. Math. Probl. Eng. 2015, 1–11.
Karabağ, O., Tan, B., 2019. Purchasing, production, and sales strategies for a production system with limited capacity, fluctuating sales and purchasing prices. IISE Trans. 51, 921–942.
Koole, G., 2006. Monotonicity in Markov reward and decision chains: Theory and applications. Found. Trends Stoch. Syst. 1, 1–76.
Liberopoulos, G., Dallery, Y., 2000. A unified framework for pull control mechanisms in multistage manufacturing systems. Ann. Oper. Res. 93, 325–355.
Liu, M., Feng, M., Wong, C.Y., 2014. Flexible service policies for a Markov inventory system with two demand classes. Int. J. Prod. Econ. 151, 180–185.
Manafzadeh Dizbin, N., Tan, B., 2019. Modelling and analysis of the impact of correlated inter-event data on production control using Markovian arrival processes. Flex. Serv. Manuf. J. 31, 1042–1076.
Manuel, P., Sivakumar, B., Arivarignan, G., 2007. A perishable inventory system with service facilities, MAP arrivals and PH — Service times. J. Syst. Sci. Syst. Eng. 16, 62–73.
Manuel, P., Sivakumar, B., Arivarignan, G., 2008. A perishable inventory system with service facilities and retrial customers. Comput. Ind. Eng. 54, 484–501.
Muharremoglu, A., Tsitsiklis, J.N., 2008. A single-unit decomposition approach to multiechelon inventory systems. Oper. Res. 56, 1089–1103.
Nasr, W.W., Maddah, B., 2015. Continuous (s, S) policy with MMPP correlated demand. European J. Oper. Res. 246, 874–885.
Neuts, M.F., 1979. A versatile Markovian point process. J. Appl. Probab. 16, 764–779.
Özekici, S., Parlar, M., 1999. Inventory models with unreliable suppliers in a random environment. Ann. Oper. Res. 91, 123–136.
Schomig, A.K., Mittler, M., 1995. Autocorrelation of cycle times in semiconductor manufacturing systems. In: Winter Simulation Conference Proceedings. pp. 865–872.
Sethi, S.P., Cheng, F., 1997. Optimality of (s,S) policies in inventory models with Markovian demand. Oper. Res. 45, 931–939.
Shanthikumar, J.G., Ding, S., Zhang, M.T., 2007. Queueing theory for semiconductor manufacturing systems: A survey and open problems. IEEE Trans. Autom. Sci. Eng. 4, 513–522.
Sharifnia, A., 1988. Production control of a manufacturing system with multiple machine states. IEEE Trans. Automat. Control 33, 620–625.
Song, J.S., Zipkin, P., 1993. Inventory control in a fluctuating demand environment. Oper. Res. 41, 351–370.
Song, J.S., Zipkin, P., 1996a. Evaluation of base-stock policies in multiechelon inventory systems with state-dependent demands: Part i: State-independent policies. Nav. Res. Logist. 39, 715–728.
Song, J.S., Zipkin, P., 1996b. Evaluation of base-stock policies in multiechelon inventory systems with state-dependent demands:part II: State-dependent depot policies. Nav. Res. Logist. 43, 381–396.
Tan, B., 2002. Production control of a pull system with production and demand uncertainty. IEEE Trans. Automat. Control 47, 779–783.
Tan, B., 2018. Production Control with Price, Cost, and Demand Uncertainty. OR Spectrum, pp. 1–29.
Tan, B., Lagershausen, S., 2017. On the output dynamics of production systems subject to blocking. IISE Trans. 49, 268–284.
Veatch, M.H., Wein, L.M., 1994. Optimal control of a two-station tandem production–inventory system. Oper. Res. 42, 337–350.
de Véricourt, F., Karaesmen, F., Dallery, Y., 2002. Optimal stock allocation for a capacitated supply system. Manage. Sci. 48, 1486–1501.
Xia, L., He, Q.-M., Alfa, A.S., 2017. Optimal control of state-dependent service rates in a MAP/M/1 queue. IEEE Trans. Automat. Control 62, 4965–4979.
Zhao, N., Lian, Z., 2011. A queueing-inventory system with two classes of customers. Int. J. Prod. Econ. 129, 225–231.