Math 20C - Fall 2007
Final - Answers

1.a

Line $\mathcal{L}_1$:

Any point (x, y, z) on the line satisfies

$$\underbrace{(x-1,\ y-1,\ z-1)}_{\substack{\text{a displacement} \\ \text{vector along the line}}} = t\,\underbrace{(1,0,1)}_{\vec{r_1}}$$

Equation for $\mathcal{L}_1$:

$x = 1+t, \quad y = 1, \quad z = 1+t$

Line $\mathcal{L}_2$:

Line is parallel to the vector

$$\vec{r_2} = (1,2,1) - (1,1,1) = (0,1,0)$$

By the same reasoning as for $\mathcal{L}_1$,
Equation for $\mathcal{L}_2$:

$x = 1, \quad y = 1+t, \quad z = t$

1.b All we need to do is to find the angle between the vectors parallel to lines.

$$\cos \underbrace{\Theta}_{\substack{\text{angle} \\ \text{between} \\ \vec{r_1} \text{ and } \vec{r_2}}} = \frac{\vec{r_1} \cdot \vec{r_2}}{|\vec{r_1}|\,|r_2|} = \frac{(1,0,1) \cdot (0,1,0)}{\sqrt{2} \cdot 1} = 0$$

Therefore $r_1 \perp r_2$, that is the angle between the lines is $\Theta = 90°$.

c. Take the cross-product to find a vector perpendicular to both r_1 and r_2

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -\vec{i} + \vec{k}$$

2. a.

$$g'(t) = f_1(u(t), v(t))\, u'(t) + f_2(u(t), v(t))\, v'(t)$$

where $f_1(u(t), v(t)) = e^{v(t)}$, $f_2(u(t), v(t)) = u(t)e^{v(t)}$
$u'(t) = -\sin(t)$, $v'(t) = \cos(t)$

$$\begin{aligned} g'(t) &= e^{v(t)}(-\sin(t)) + u(t)e^{v(t)}\cos(t) \\ &= e^{\sin(t)}(-\sin(t)) + \cos(t)e^{\sin(t)}\cos(t) \\ &= e^{\sin(t)}(-\sin(t) + \cos^2(t)) \end{aligned}$$

b. (*) $h_1(x,y) = f_1(u,v)\cancel{\frac{\partial u}{\partial x}} + f_2(u,v)\cancel{\frac{\partial v}{\partial x}}\, v'(x)$

(**) $h_2(x,y) = f_1(u,v)\cancel{\frac{\partial u}{\partial y}}^{u'(y)} + f_2(u,v)\cancel{\frac{\partial v}{\partial y}}$

where

$$f_1(u,v) = \frac{\cos u}{\cos v}, \quad f_2(u,v) = \frac{\sin u \sin v}{\cos^2 v}$$

$$v'(x) = 2x \quad \text{and} \quad u'(y) = -\frac{2}{y^3}$$

Plug these values in (*) and (**)

$$h_1(x,y) = \frac{(\sin u)(\sin v)}{\cos^2 v}\, 2x = \frac{\sin(\frac{1}{y^2})\sin(x^2)}{\cos^2(x^2)}\, 2x$$

$$h_2(x,y) = \frac{\cos u}{\cos v}\left(-\frac{2}{y^3}\right) = \frac{\cos(1/y^2)}{\cos(x^2)}\left(-\frac{2}{y^3}\right)$$

3. The problem we need to solve

MINIMIZE $S(a,b,h)$

subject to $V(a,b,h)=500$

where

$S(a,b,h) = ab + 2ah + 2bh$ (The surface area of the box)

$V(a,b,h) = abh$ (volume of the box)

We can eliminate h using the constraint $V(a,b,h) = abh = 500$ and obtain the unconstrained optimization problem

MINIMIZE $\overbrace{ab + 2\cancel{a}\frac{500}{\cancel{a}b} + 2\cancel{b}\frac{500}{a\cancel{b}}}^{f(a,b)}$

and

$$h = \frac{500}{ab}$$

We need The critical points of f,

$$\nabla f(a,b) = \left(b - \frac{1000}{a^2},\ a - \frac{1000}{b^2}\right) = (0,0)$$

$\Longrightarrow$ (1) $b = \dfrac{1000}{a^2}$ and (2) $a = \dfrac{1000}{b^2}$

Substitute $a = \dfrac{1000}{b^2}$ in (1)

$$b = \frac{1000}{\left(\frac{10^6}{b^4}\right)} \Longrightarrow b^3 = 1000 \Longrightarrow b = 10$$

$$a = \frac{1000}{b^2} = \frac{1000}{100} = 10$$

Using $h = \dfrac{500}{ab}$

$$h = \frac{500}{(10)(10)} = 5$$

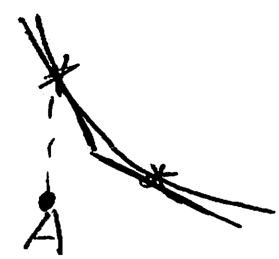
$\boxed{\text{The dimensions minimizing the surface area } a = b = 10,\ h = 5}$

4

a) $f_x(x,y) < 0$ at A, because f is decreasing in the positive x direction.

$f_y(x,y) > 0$ at A, because f is increasing in the positive y-direction

$f_{xx}(x,y) > 0$ at A, because the function is becoming flat in the positive x-direction, that is in absolute value tangent slopes are decreasing. But the tangent slopes are negative (since $f_x(x,y) < 0$), that is tangent slopes are increasing



Slopes are negative, but the function is becoming flat. Therefore the slopes are increasing.

b. At B

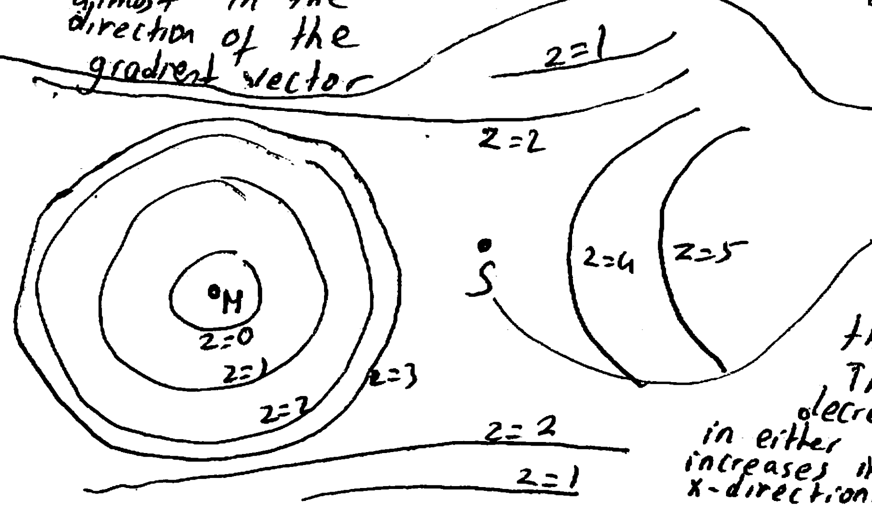
$$D_u f(x,y) \geq D_v f(x,y) \geq D_w f(x,y) \cong 0$$

$D_u f(x,y) \cong |\nabla f(x,y)|$

u points almost in the direction of the gradient vector

w points almost in the direction of the tangent vector

c.



At S, the function is flat possibly the gradient is $\mathbf{0}$. The function decreases vertically in either direction and increases in the positive x-direction.

4

5.

$$\text{Area of the Archimedean Spiral} = \overbrace{\iint_D dx\,dy}^{\text{In Eucledian coordinates}} = \overbrace{\iint_D r\,dr\,d\theta}^{\text{In polar coordinates}}$$

where D is the area enclosed by the Archimedean spiral. In polar coordmates

$$D = \{(r,\theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1+\theta\}$$

Therefore

$$\text{Area of the Archimedean spiral} = \int_0^{2\pi}\int_0^{1+\theta} r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}\Big|_{r=0}^{1+\theta}\right)d\theta$$

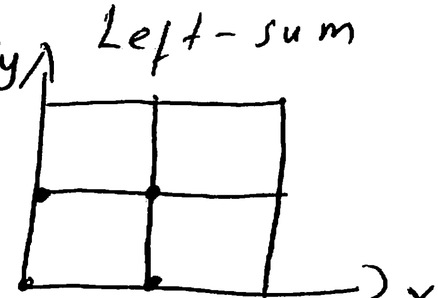
$$= \frac{1}{2}\int_0^{2\pi}(1+\theta)^2\,d\theta = \frac{1}{2}\,\frac{(1+\theta)^3}{3}\Big|_{\theta=0}^{2\pi}$$

$$= \frac{1}{6}\left((2\pi+1)^3 - 1\right)$$

$$= \frac{1}{6}\left(8\pi^3 + 12\pi^2 + 6\pi\right)$$

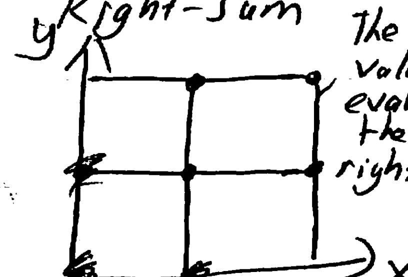
6.

a. Left-sum



The function values are evaluated at the lower-left corners

Right-sum



The function values are evaluated at the upper right corners

Left-sum $= f(0,0)\Delta x\Delta y + f(1,0)\Delta x\Delta y + f(0,1)\Delta x\Delta y + f(1,1)\Delta x\Delta y$

Right-sum $= f(1,1)\Delta x\Delta y + f(1,2)\Delta x\Delta y + f(2,1)\Delta x\Delta y + f(2,2)\Delta x\Delta y$

$f(0,0)\quad f(1,0),\quad f(0,1)\quad f(1,1)\qquad \Delta x,\ \Delta y$

$$\text{Left-sum} = (9+8+8+7)(1)(1) = 32$$
$$\text{Right-sum} = (7+4+4+1)\ (1)(1) = 16$$

$f(0,0)\quad f(1,0)\quad f(0,1)\quad f(1,1)$

b. The function is decreasing in D. It will take the largest value in a rectangle at the lower-left corner and the smallest value at the upper-right corner. Therefore

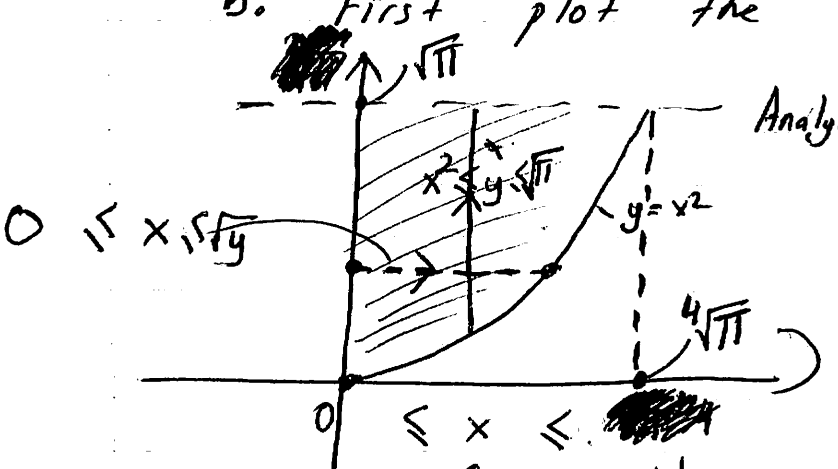
RIGHT SUM $\leq$ EXACT VALUE $\leq$ LEFT SUM

7. a)

$$\int_1^3\int_0^{\pi} x\sin(y)\,dy\,dx = \int_1^3 \left(-x\cos y\right)\Big|_{y=0}^{\pi} dx$$

$$= \int_1^3 -x\,\underbrace{(\cos(\pi)-\cos(0))}_{-2}\,dx = x^2\Big|_{x=1}^{3} = 8$$

b. First plot the domain in 2D space



Analytic Integration with respect to y is impossible. Because we don't know an anti-derivative for $\sin(y^2)$

So Let's find the x-simple representation for the domain

$$\overbrace{D = \{(x,y) : 0 \leq y \leq \sqrt{\pi},\ 0 \leq x \leq \sqrt{y}\}}^{x\text{-simple}}$$

Changing the order of integration yields

$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} x \sin(y^2)\, dy\, dx = \int_0^{\sqrt{\pi}} \int_0^{\sqrt{y}} x(\sin(y^2))\, dx\, dy$$

$$= \int_0^{\sqrt{\pi}} \left(\frac{x^2}{2} \sin(y^2)\right)\Bigg|_{x=0}^{\sqrt{y}} dy = \int_0^{\sqrt{\pi}} \frac{y}{2} \sin(y^2)\, dy$$

$$= -\frac{1}{4} \cos(y^2)\Bigg|_{y=0}^{\sqrt{\pi}}$$

$$= -\frac{1}{4} \cos(\pi) + \frac{1}{4} = \frac{1}{2}$$

8.

$$M = \int_0^{\pi/2}\int_0^{1}\int_0^{1} y \cos(x) e^z \, dz \, dy \, dx$$

$$= \int_0^{\pi/2}\int_0^{1} y \cos(x) e^z \Big|_{z=0}^{1} dy\,dx = \int_0^{\pi/2}\int_0^{1} y \cos(x)(e-1)\,dy\,dx$$

$$= \int_0^{\pi/2} (e-1)\frac{y^2}{2}\cos(x)\Big|_{y=0}^{1} dx = \int_0^{\pi/2} \frac{(e-1)}{2}\cos(x)\,dx$$

$$= \frac{e-1}{2}\sin(x)\Big|_{x=0}^{\pi/2} = \frac{e-1}{2}$$