Homework 1 - Answers to selected questions

1.

a. Since it is the plane parallel to the xz-plane, the y-component is fixed but the x, z components are arbitrary (and equal to -2)

The set of points on the plane $= \{ (x,y,z) : y = -2 \}$

The equation for the plane is $y = -2$

that is as long as $y=-2$, the point (x,y,z) lies on the plane.

b. First assume you are in 2D and your coordinate axes are x and y. The line with x-intercept and y-intercept one passes through

$$\underset{x_1\ \ y_1}{(1, 0)} \quad \text{and} \quad \underset{x_2\ \ y_2}{(0, 1)}$$

$m =$ Slope of the line $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-0}{0-1} = -1$

$y_0 =$ y-intercept $= 1$

~~$y - y_0 = m(x$~~

$$y = mx + y_0$$

$$(*)\ y = -x + 1$$

Now in 3D, since the plane is parallel to the z-axis, regardless of the z-component a point (x,y,z) is on the plane as long as its [redacted] its (x,y) components satisfy $(*)$.

The equation for the plane is $y = -x + 1$

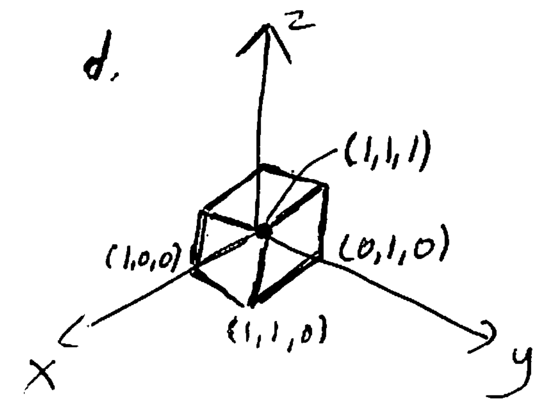
c. Again assume you are in 2D with y, z as your coordinate axes. The equation of the circle of radius 2 centered at $(3,1)$ is

$$(**) \quad (x-3)^2 + (y-1)^2 = 4$$

In 3D the cylinder is extending infinitely parallel to the z-axis. Therefore regardless of the z value, the point (x,y,z) is on the cylinder as long x, y components satisfy $(**)$. The equation for the cylinder is

$$\boxed{(x-3)^2 + (y-1)^2 = 4}$$

d.



A point is inside ~~[illegible]~~ or on the unit cube in the figure if

$$\boxed{\begin{array}{c} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ 0 \leq z \leq 1 \end{array}}$$

4.

a. Use the geometric definition of the dot product.

$$v \cdot 2v = |v|\,|2v| \cos 0 \quad \text{since the angle between } \vec{v} \text{ and } 2v \text{ is } 0$$
$$= \boxed{2|v|^2}$$

b. Again using the geometric definition

$$v \cdot 2w = |v|\,|2w| \cos 90 \quad \text{since } v \perp 2w$$
$$= \boxed{0}$$

c. (***) $(v+w)\cdot(v-w) = v\cdot v + v\cdot w$
$+ w\cdot v - w\cdot w$

Above we use the properties that the dot-product distributes over addition and $(c_2 v)\cdot(c_1 w) = c_1 c_2 (v\cdot w)$.

The dot-product is commutative, that is
$$v\cdot w = w\cdot v$$
Therefore the right-hand side of (***) simplifies
$$(v+w)\cdot(v-w) = v\cdot v - w\cdot w$$
$$= |v||v|\cos 0 - |w||w|\cos 0$$
$$= \boxed{|v|^2 - |w|^2}$$

5. a. To find the cosine of the angle between u and v, equate the algebraic and geometric definitions of the dot-product.

$$u\cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\,|v| \cos\theta$$ angle between

$$\Rightarrow \cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|u|.|v|} = \frac{1(-1) + 2(1) + 0(-2)}{\sqrt{5}\sqrt{6}}$$
$$= \boxed{\frac{1}{\sqrt{30}}}$$

b. According to the geometric definition

$$u\cdot w = |u|\underbrace{|w|}_{\text{since } w \text{ is a unit vector}}\cos\theta = |u|\cos\theta$$

is maximized when $\theta = 0$, that is when u and w point in the same direction

w = unit vector pointing in the direction of u $= \dfrac{u}{|u|} = \dfrac{\vec{i}+2\vec{j}}{\sqrt{1+4}} = \boxed{\dfrac{1}{\sqrt{5}}\vec{i} + \dfrac{2}{\sqrt{5}}\vec{j}}$

c. Again using the geometric definition

$$v \cdot w = |v|\,|w| \cos\theta = |v| \cos\theta$$

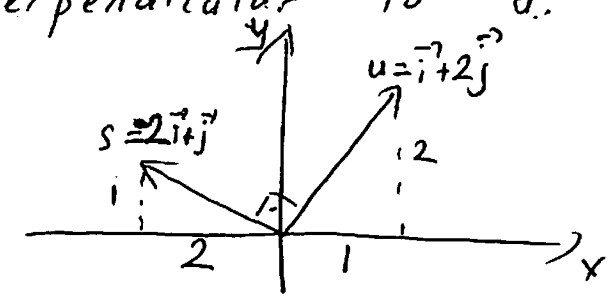
is minimized when $\theta = \pi$ ($\cos\theta' \geq \cos\pi = -1$ for all θ'), that is w points in the direction opposite to v.

$$w = \underbrace{\frac{-v}{|v|}}_{\text{unit vector opposite to } v} = \frac{-(j-k)}{\sqrt{1+1}} = \boxed{\frac{-1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k}$$

d. Since u is on the xy-plane any vector along the z-axis is perpendicular to u, for instance

$$s = \vec{k} \quad \text{and} \quad s = -2\vec{k}$$

There are also vectors on the xy-plane perpendicular to u.



u has slope 2

any vector with slope equal to $-\frac{1}{2}$ must be perpendicular to u, for instance

$$\boxed{\vec{s} = 2\vec{i} - \vec{j} \quad \text{or} \quad \vec{s} = -2\vec{i} + \vec{j}.}$$

6.

a. You can find a vector perpendicular to a pair of vectors using the cross-product.

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \boxed{-2\vec{i} + \vec{j} + \vec{k}}$$

b. You need to use the properties of the cross-product. In general

(1) $p \times (q+r) = p \times q + p \times r$

$(q+r) \times p = q \times p + r \times p$

(2) $(c_1 p) \times (c_2 q) = (c_1 c_2)(p \times q)$

(3) $p \times q = -q \times p$

From the first two properties

$$(u+v) \times (u-v) = \underbrace{u \times u}_{0} - u \times v + v \times u + \underbrace{v \times v}_{0}$$

0 (since $p \times q = 0$ when p and q point in the same or opposite direction)

Using the third property

$$\boxed{\begin{aligned}(u+v) \times (u-v) &= -2(u \times v) = -2(-2\vec{i} + \vec{j} + \vec{k}) \\ &= 4\vec{i} - 2\vec{j} - 2\vec{k}\end{aligned}}$$

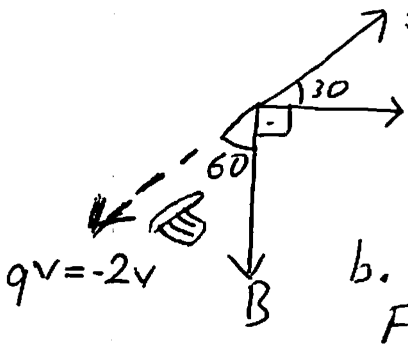
The dot product distributes over addition

$$\boxed{(u+v) \cdot (u \times v) = \underbrace{u \cdot (u \times v)}_{0} + \underbrace{v \cdot (u \times v)}_{0} = 0}$$

since $u \perp u \times v$ and $v \perp u \times v$

7. a. Use the geometric definition of the cross-product

$|F| = |qv \times B| = |qv||B| \sin\theta$ angle between qv and B



$= |q|\,|v|\,|B| \sin 60$

$= 2 \cdot 5 \cdot 15 \frac{\sqrt{3}}{2} = 75\sqrt{3}$ Newtons

b. Using the right-hand rule $F = qv \times B$ points outside the page (containing $\vec{B}$ and $\vec{v}$).

8. For the first line
$r_1 = (1,2,1)$ is parallel to the line
and $P_1 = (-1,-2,-1)$ is on the line

pick a point $P=(x,y,z)$ on the line.

$\underbrace{(x+1, y+2, z+1)}_{\overrightarrow{P_1P}} = t(1,2,1)$

$\overrightarrow{P_1P} = t r_1$ (r_1 and $\overrightarrow{P_1P}$ are parallel)

$\boxed{\text{The parametric equation for } \mathcal{L}_1 \text{ ~~the product line~~: } x = t-1,\ y = 2t-2,\ z = t-1}$

For the second line
$r_2 = (3,3,3) - (-1,1,-1) = (4,2,4)$ is parallel to the line
$P_1 = (-1,1,-1)$ is on the line

A point $P=(x,y,z)$ on the line must satisfy
$(x+1, y-1, z+1) = s(4,2,4)$

$\boxed{\text{The parametric equation for } \mathcal{L}_2\text{: } x = 4s-1,\ y = 2s+1,\ z = 4s-1}$

b. Eliminate the variables t and s in the parametric equations

For $\mathcal{L}_1$, $t = x+1 = \frac{y+2}{2} = z+1$

$$\boxed{x+1 = \frac{y+2}{2} = z+1 \text{ : Symmetric equation for } \mathcal{L}_1}$$

For $\mathcal{L}_2$, $s = \frac{x+1}{4} = \frac{y-1}{2} = \frac{z+1}{4}$

$$\boxed{\frac{x+1}{4} = \frac{y-1}{2} = \frac{z+1}{4} \text{ : Symmetric equation for } \mathcal{L}_2}$$

c. Use the parametric equations. Equate the corresponding components.

(1) $t-1 = 4s-1$, (2) $2t-2 = 2s+1$, (3) $t-1 = 4s-1$

From (1) $t = 4s$. Plug this equation in (2)

$8s-2 = 2s+1 \Rightarrow 6s = 3 \Rightarrow s = \frac{1}{2}$ and $t = 4s = 2$

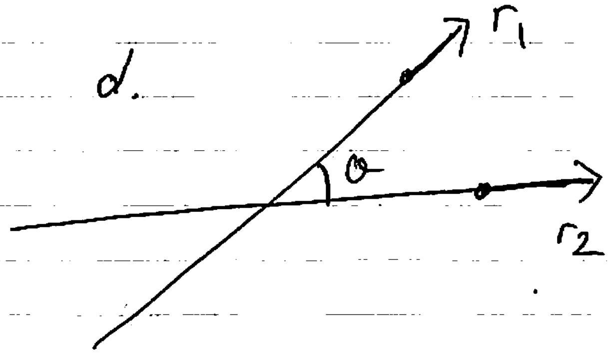
At $t=2$, the point on $\mathcal{L}_1$

$x = 2-1 = 1$, $y = 2(2)-2 = 2$, $z = 2-1 = 1$

At $s = 1/2$, the point on $\mathcal{L}_2$

$x = 4s-1 = 4(1/2)-1 = 1$, $y = 2(1/2)+1 = 2$, $z = 4s-1 = 4(\frac{1}{2})-1 = 1$

Two lines intersect at $(1,2,1)$

d. 

The angle between the direction vectors r_1, r_2 gives the angle between lines.

$$r_1 \cdot r_2 = (1,2,1) \cdot (4,2,4)$$
$$= 4+4+4 = 12 = |r_1|\,|r_2| \cos\theta$$
$$= \underbrace{\sqrt{1^2+2^2+1^2}}_{\sqrt{6}}\ \underbrace{\sqrt{4^2+2^2+4^2}}_{6} \cos\theta$$

$$\cos\theta = 2/\sqrt{6} \Rightarrow \quad \theta \cong \pi/5$$