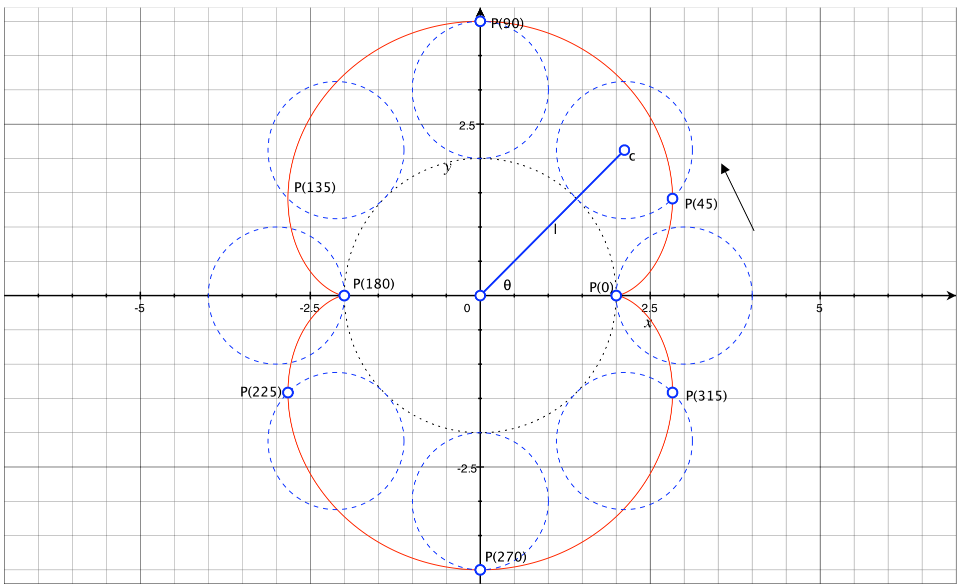
Math 20C (Lecture C) - Homework 2

(Due on November 5th, Monday by 3pm)

This homework covers sections 10.1-10.2, 13.1-4 from Stewart's book. You need to return only the questions marked with (∗). The others are provided for practicing purposes.

1.(*) (Section 10.1) Consider two circles that are placed next to each other initially.

- The circle $\mathcal{C}_1$ (dotted curve in the figure) is centered at the origin and of radius two.
- The circle $\mathcal{C}_2$ (dashed curve in the figure) is centered at $(3, 0)$ and of radius one.



The circle $\mathcal{C}_2$ is rolled over $\mathcal{C}_1$ in the counter clockwise direction until it reaches its original position. The trajectory of the point (the solid curve in the figure) on $\mathcal{C}_2$ initially located at $P(0) = (2, 0)$ (where two circles touch each other initially) is kidney-shaped and called *nephroid*. Show that a parametric equation for the nephroid is

$$P(\theta) = (x(\theta), y(\theta)) = (3\cos(\theta) - cos(3\theta), 3\sin(\theta) - \sin(3\theta))$$

where θ is the angle between the line l joining the centers of the circles and the horizontal axis. It suffices to derive the parametric equation for $0 \leq \theta \leq \frac{\pi}{6}$.

2. (Section 10.2) For the nephroid in question **1.** determine the leftmost, rightmost, uppermost and lowermost points on the curve.

3. (Section 10.2) Find the area enclosed by the nephroid

$$x(\theta) = 3\cos(\theta) - cos(3\theta) \text{ and } y(\theta) = 3\sin(\theta) - \sin(3\theta), \quad \text{where } 0 \leq \theta < 2\pi.$$

(Hint : Use integration by parts and the identity $\sin(3\theta) = \sin(\theta)\cos(2\theta) + \cos(\theta)\sin(2\theta)$.)

4.(*) (Section 13.1) Consider the vector-valued function

$$\mathbf{r}(t) = \left(\arctan(t), e^{-t}, \frac{\ln(t)}{t-3}\right)$$

a. Find the domain and range of $\mathbf{r}(t)$.

b. Find $\lim_{t\to\infty} \mathbf{r}(t)$.

5.(*) (Section 13.2) Find the equations of the lines tangent to the parametric curve

$$x(t) = \cos(t) + t \text{ and } y(t) = \sin(t) + t, \quad t \geq 0$$

with slope one. (Remark : there are infinitely many points on the curve where the slopes of the tangent lines equal to one. However, there are *only two tangent lines with slope one.*)

6. (Section 13.3) The parametric curve represented by the parametric equation

$$x(t) = t\cos(t), \quad y(t) = t\sin(t) \text{ and } z(t) = \frac{2\sqrt{2}t\sqrt{t}}{3} \quad t \geq 0$$

is a *helix.*

a. Plot the helix in 3D space. Indicate the direction of the curve. Label some of the points on the helix with the associated t value.

b. Find the arc-length $s(t')$ of the spiral from $t = 0$ to $t = t'$.

c. Parametrize the helix in terms of the arc-length s.

7.(*) (Section 13.3) Show that the curvature of the *spiral* with parametric equation

$$x(t) = t\cos(t) \text{ and } y(t) = t\sin(t), \quad t \geq 0$$

approaches 0 as $t \to \infty$.

8.(*) (Section 13.3) The parametric curve given by the equations

$$x(t) = \cos(t), \quad y(t) = \sin(t) \;\text{ and }\; z(t) = t, \quad t \geq 0$$

evolves on a cylinder extending along the z-axis.

a. Find the arc-length $s(t')$ of the curve from $t = 0$ to $t = t'$.

b. Find the unit tangent vector $\hat{\mathbf{T}}(s)$ in terms of the parameter s.

c. Show that the curvature is constant and equal to 1/2.

d. Show that the unit normal vector $\hat{\mathbf{N}}(s)$ is parallel to the xy-plane and points towards the z-axis at all s.

e. Determine the unit binormal vector $\hat{\mathbf{B}}(\mathbf{s})$.

f. Find the equation of the osculating plane at $s = \frac{\pi}{\sqrt{2}}$.

9. (Section 13.4 - modified from exercise 13.4.20 from Stewart's book)
Let $\mathbf{r}(t) = e^{-t}\vec{i} + e^{-2t}\vec{j} + t\vec{k}$ be the position of an object of mass m.

a. Find the velocity, speed and acceleration of the object.

b. What is the force required as a function of time so that the object moves in the trajectory $\mathbf{r}(t)$?

c. Describe in words the movement of the object. Is it moving faster or slower in time? How does the acceleration vary in time?