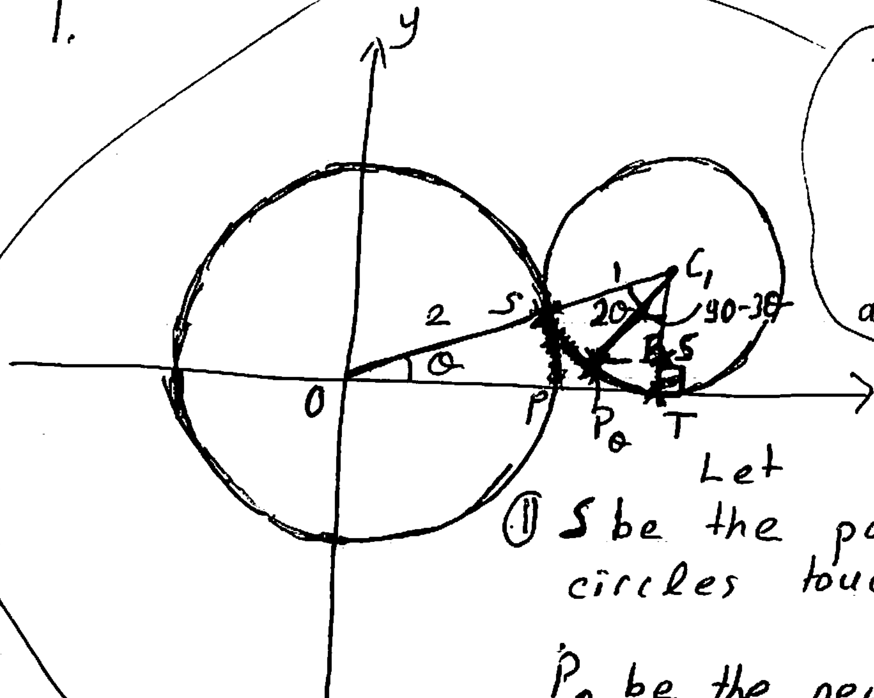
Homework 2 - Answers to selected questions

1.



The arcs on the circles from S to P and from S to P_θ are of equal length

Let

① S be the point where two circles touch

P_θ be the new position of the point P.

T be the point where the vertical line from the center of the nonstationary circle intersects the horizontal axis

C_1 be the center of the non-stationary circle.

S be the point where the horizontal line from P_θ intersect the vertical line passing through C_1.

②

length of the arc on the stationary circle from S to P

=

2θ

=

length of the arc on the non-stationary circle from S to P_θ

=

$1 \widehat{SC_1P_\theta}$

angle between the lines segments SC_1 and $P_\theta C_1$

Therefore

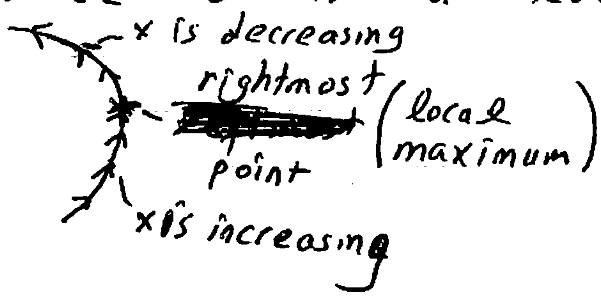
$\widehat{SC_1P_\theta} = 2\theta$

We need to find the coordinates of $P_\theta = (x(\theta), y(\theta))$ as a function of θ

$$x(\theta) = |OT| - |P_\theta T| = 3\cos\theta - 1\sin(90-3\theta))$$
$$= 3\cos\theta - \cos 3\theta$$

$$y(\theta) = |C_1 T| - |C_1 S| = 3\sin\theta - 1\cos(90-3\theta)$$
$$= 3\sin\theta - \sin 3\theta$$

2. At the leftmost or righmost point
$x'(\theta) = 0$
since θ is a local minimum or maximum.



$$x'(\theta) = -3\sin\theta + 3\sin 3\theta$$
$$x'(\theta) = 0 \Longrightarrow \cancel{3}\sin 3\theta = \cancel{3}\sin\theta$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
are the local extremum of $x(\theta)$

$$x\left(\frac{\pi}{4}\right) = x\left(\frac{7\pi}{4}\right) = 3\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}$$
$$= 3\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

~~are the rightmost~~

$$x\left(\frac{3\pi}{4}\right) = x\left(\frac{5\pi}{4}\right) = 3\cos\frac{3\pi}{4} - \cos\frac{9\pi}{4}$$
$$= -\frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\sqrt{2}$$

$(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (2\sqrt{2}, \sqrt{2})$
$(x(\frac{7\pi}{4}), y(\frac{7\pi}{4})) = (2\sqrt{2}, -\sqrt{2})$ are the righmost points.

$(x(\frac{3\pi}{4}), y(\frac{3\pi}{4})) = (-2\sqrt{2}, \sqrt{2})$
$(x(\frac{5\pi}{4}), y(\frac{5\pi}{4})) = (-2\sqrt{2}, -\sqrt{2})$ are the leftmost points.

At the uppermost or lowermost points

$$y'(\theta) = 3\cos\theta - 3\cos 3\theta = 0$$

$$\Longrightarrow \cos\theta = \cos 3\theta$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ are local extremum of $y(\theta)$.

$$y(\tfrac{\pi}{2}) = 3\sin\tfrac{\pi}{2} - \sin\tfrac{3\pi}{2} = 4$$

$$y(\tfrac{3\pi}{2}) = 3\sin\tfrac{3\pi}{2} - \sin\tfrac{9\pi}{2} = -4$$

$(x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (0, 4)$ is the uppermost point

$(x(\frac{3\pi}{2}), y(\frac{3\pi}{2})) = (0, -4)$ is the lowermost point

4. a.

$\arctan(t)$ is defined for all t
e^{-t} is defined for all t
$\dfrac{\ln t}{t-3}$ is defined if $t>0$ and $t\neq 3$

$$\boxed{\begin{aligned}\text{Domain} &= \{t \in \mathbb{R} : t>0,\ t\neq 3\}\\ &= (0,3)\cup(3,\infty)\end{aligned}}$$

$$\boxed{\text{Range} = \{(\arctan(t), e^{-t}, \ln t/(t-3)) : t>0 \text{ and } t\neq 3\}}$$

b. $$\lim_{t\to\infty}\left(\arctan(t), e^{-t}, \frac{\ln t}{t-3}\right)$$

$$= \left(\lim_{t\to\infty}\arctan(t), \lim_{t\to\infty} e^{-t}, \lim_{t\to\infty}\frac{\ln t}{t-3}\right)$$

$$= \boxed{\left(\frac{\pi}{2}, 0, 0\right)}$$

goes to 0, since $\ln t$ grows slower than $t-3$

5. We are looking for the points where

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t+1}{-\sin t+1} = 1,$$

or equivalently t values satisfying
$\cos t+1 = -\sin t+1 \implies \tan t = -1.$

Slope of the tangent line is 1 at
$t = \dfrac{3\pi}{4} + 2\pi k$, for any integer k
$t = \dfrac{7\pi}{4} + 2\pi k$

$$x\left(\frac{3\pi}{4}+2\pi k\right)=\cos\left(\frac{3\pi}{4}+2\pi k\right)+\frac{3\pi}{4}+2\pi k$$
$$=-\frac{1}{\sqrt{2}}+\frac{3\pi}{4}+2\pi k$$
$$y\left(\frac{3\pi}{4}+2\pi k\right)=\sin\left(\frac{3\pi}{4}+2\pi k\right)+\frac{3\pi}{4}+2\pi k$$
$$=\frac{1}{\sqrt{2}}+\frac{3\pi}{4}+2\pi k$$

The equation of the tangent line at $t=\frac{3\pi}{4}+2\pi k$

$$y-y\left(\frac{3\pi}{4}+2\pi k\right)=\left.\frac{dy}{dx}\right|_{t=\frac{3\pi}{4}+2\pi k}\left(x-x\left(\frac{3\pi}{4}+2\pi k\right)\right)$$
$$y-\left(\frac{1}{\sqrt{2}}+\frac{3\pi}{4}+2\pi k\right)=x-\left(-\frac{1}{\sqrt{2}}+\frac{3\pi}{4}+2\pi k\right)$$
$$\boxed{y=x+\sqrt{2}}$$

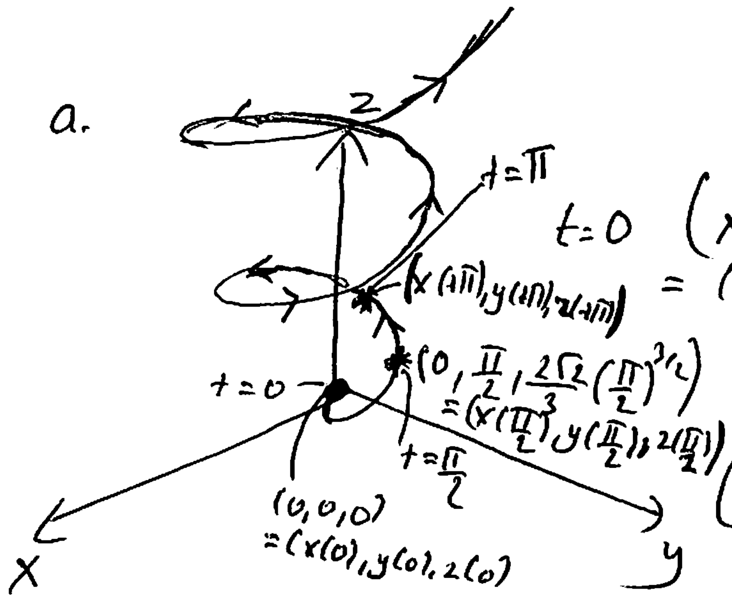
$$x\left(\frac{7\pi}{4}+2\pi k\right)=\cos\left(\frac{7\pi}{4}+2\pi k\right)+\frac{7\pi}{4}+2\pi k$$
$$=\frac{1}{\sqrt{2}}+\frac{7\pi}{4}+2\pi k$$
$$y\left(\frac{7\pi}{4}+2\pi k\right)=\sin\left(\frac{7\pi}{4}+2\pi k\right)+\frac{7\pi}{4}+2\pi k$$
$$=-\frac{1}{\sqrt{2}}+\frac{7\pi}{4}+2\pi k$$

The equation of the tangent line at $t=\frac{7\pi}{4}+2\pi k$

$$y-y\left(\frac{7\pi}{4}+2\pi k\right)=\left.\frac{dy}{dx}\right|_{t=\frac{7\pi}{4}+2\pi k}\left(x-x\left(\frac{7\pi}{4}+2\pi k\right)\right)$$
$$y-\left(-\frac{1}{\sqrt{2}}+\frac{7\pi}{4}+2\pi k\right)=x-\left(\frac{1}{\sqrt{2}}+\frac{7\pi}{4}+2\pi k\right)$$
$$\boxed{y=x-\sqrt{2}}$$

6.

a.



$t=0$ $(x(0), y(0), z(0)) = (0,0,0)$

$t = \frac{\pi}{2}$ $(x(\frac{\pi}{2}), y(\frac{\pi}{2}), z(\frac{\pi}{2})) = (0, \frac{\pi}{2}, \frac{2\sqrt{2}}{3}(\frac{\pi}{2})^{3/2})$

$t=\pi$
$(x(\pi), y(\pi), z(\pi)) = (-\pi, 0, \frac{2\sqrt{2}}{3}(\pi)^{3/2})$

b. $s = f(t') = \int_0^{t'} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$

where
$x'(t) = \cos t - t\sin t$ $y'(t) = \sin t + t\cos t$ $z'(t) = \frac{2\sqrt{2}}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}\sqrt{t}$

$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t}$
$= \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\cos t\sin t} + t^2\cos^2 t + 2t}$
$= t+1$

Therefore
$$\boxed{s = f(t') = \int_0^{t'} t+1\, dt = \frac{(t')^2}{2} + t'}$$

c.

$$s = \frac{(t')^2}{2} + t' = \left(\frac{t'}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}$$

Leave t' alone on the right

$$\sqrt{2}\sqrt{\left(s+\frac{1}{2}\right)} - \frac{1}{\sqrt{2}} = t' = f^{-1}(s)$$

The relation between $\hat{r}(s)$ and $r(t)$ is as follows.

$$r(t) = r(f^{-1}(s)) = \hat{r}(s)$$

Therefore

$$\hat{r}(s) = r\left(\sqrt{2s+1} - 1\right)$$

$$= \boxed{\left((\sqrt{2s+1}-1)\cos(\sqrt{2s+1}-1),\ (\sqrt{2s+1}-1)\sin(\sqrt{2s+1}-1),\ \frac{2\sqrt{2}}{3}(\sqrt{2s+1}-1)^{3/2}\right)}$$

7. Use the formula

$$\kappa = \frac{|T'(t)|}{|r'(t)|}$$

Note that

$$r'(t) = (\cos t - t\sin t,\ \sin t + t\cos t)$$

$$|r'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

$$= \sqrt{\cos^2 t + t^2\sin^2 t \cancel{- 2t\cos t\sin t} + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t}$$

$$= \sqrt{1+t^2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{1+t^2}}(\cos t - t\sin t,\ \sin t + t\cos t)$$

$$T'(t) = \frac{-2t}{2(1+t^2)^{3/2}}(\cos t - t\sin t, \sin t + t\cos t) + \frac{1}{\sqrt{1+t^2}}(-\sin t - \sin t - t\cos t, \cos t + \cos t - t\sin t)$$

$$\frac{T'(t)}{r'(t)} = \frac{-t}{(1+t^2)^2}(\cos t - t\sin t, \sin t + t\cos t) + \frac{1}{1+t^2}(-2\sin t - t\cos t, 2\cos t - t\sin t)$$

$$\frac{-t(1+t)}{(1+t^2)^2} \leq \underbrace{\frac{-t}{(1+t^2)^2}(\cos t - t\sin t)}_{f_1(t)} \leq \frac{t(1+t)}{(1+t^2)^2}$$

By the squeeze theorem

$$\underbrace{\lim_{t\to\infty} \frac{-t(1+t)}{(1+t^2)^2}}_{0} \leq \lim_{t\to\infty} f_1(t) \leq \underbrace{\lim_{t\to\infty} \frac{t(1+t)}{(1+t^2)^2}}_{0}$$

$$\lim_{t\to\infty} f_1(t) = 0$$

Similarly

$$\frac{-t(1+t)}{1+t^2} \leq \underbrace{\frac{-t(\sin t + t\cos t)}{(1+t^2)^2}}_{f_2(t)} \leq \frac{t(1+t)}{(1+t^2)^2}$$

approaches 0 as $t\to\infty$

approaches 0 as $t\to\infty$

$$\frac{-2-t}{1+t^2} \leq \underbrace{\frac{-2\sin t - t\cos t}{1+t^2}}_{f_3(t)} \leq \frac{2+t}{1+t^2}$$

$$\frac{-2-t}{1+t^2} \leq \underbrace{\frac{2\cos t - t\sin t}{1+t^2}}_{f_4(t)} \leq \frac{2+t}{1+t^2}$$

By the squeeze theorem

$$\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} f_2(t) = \lim_{t\to\infty} f_3(t) = \lim_{t\to\infty} f_4(t) = 0$$

$$\lim_{t\to\infty} \frac{T'(t)}{r'(t)} = \left(\lim_{t\to\infty} f_1(t)+f_3(t), \lim_{t\to\infty} f_2(t)+f_4(t)\right)$$

$$= (0,0)$$

implying

$$0 = \left|\lim_{t\to\infty} \frac{T'(t)}{r'(t)}\right| = \lim_{t\to\infty} \left|\frac{T'(t)}{r'(t)}\right|$$

8. a. $$s = f(t') = \int_0^{t'} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

where
$x'(t) = -\sin(t)$ $y'(t) = \cos t$ and $z'(t) = 1$
and

$$|r'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Therefore

$$\boxed{s = f(t') = \int_0^{t'} \sqrt{2}\, dt = \sqrt{2}\, t'}$$

b. The relation between $T(t)$ and $\hat{T}(s)$ is given by
$T(t) = T(f^{-1}(s)) = \hat{T}(s)$
with
$t = f^{-1}(s) = \frac{s}{\sqrt{2}}$.

Therefore

$$\hat{T}(s) = T\left(\frac{s}{\sqrt{2}}\right).$$

We need to find $T(t)$.

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$

$$\boxed{\hat{T}(s) = T\left(\frac{s}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right)}$$

c. By definition $\kappa(s) = |\hat{T}'(s)|$

$$= \left|\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, 0\right)\right|$$

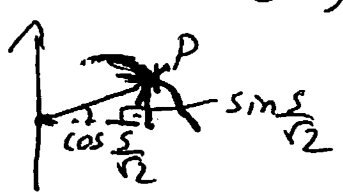
$$= \sqrt{\left(\frac{1}{2}\right)^2\cos^2\frac{s}{\sqrt{2}} + \left(\frac{1}{2}\right)^2\sin^2\frac{s}{\sqrt{2}}} = \frac{1}{2}$$

d. $$\hat{N}(s) = \frac{\hat{T}'(s)}{|\hat{T}'(s)|} = \frac{\left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right)}{1/2}$$

$$= \boxed{\left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)}$$

As the third component is 0, $\hat{N}(s)$ is parallel to the xy-plane

$\hat{K}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right)$ points towards the point P



$\hat{N}(s) = -\hat{K}(s)$ points towards the origin from the point.

e. By definition

$$\hat{B}(s) = \hat{T}(s) \times \hat{N}(s)$$

$$= \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \times \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\hat{B}(s) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} & -\sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix} = \vec{i}\left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}\right) - \vec{j}\left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\right) + \vec{k}\left(+\frac{1}{\sqrt{2}}\sin^2\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos^2\frac{s}{\sqrt{2}}\right)$$

$$= \boxed{\vec{i}\,\frac{\sin s/\sqrt{2}}{\sqrt{2}} - \vec{j}\,\frac{\cos s/\sqrt{2}}{\sqrt{2}} + \vec{k}\,\frac{1}{\sqrt{2}}}$$

f. Osculating plane contains $\hat{T}\left(\frac{\pi}{\sqrt{2}}\right)$ and $\hat{N}\left(\frac{\pi}{\sqrt{2}}\right)$.
Therefore

$$n = \hat{B}\left(\frac{\pi}{\sqrt{2}}\right) = \vec{i}\,\frac{1}{\sqrt{2}} + \vec{k}\,\frac{1}{\sqrt{2}}$$

is a normal to the plane

The point $\hat{r}\left(\frac{\pi}{\sqrt{2}}\right)$ is on the osculating plane

$$r(t) = r(f^{-1}(s)) = r\left(\frac{s}{\sqrt{2}}\right) = \hat{r}(s)$$

$$\Longrightarrow \hat{r}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

$$\hat{r}\left(\frac{\pi}{\sqrt{2}}\right) = \left(0, 1, \frac{\pi}{2}\right)$$

The equation of the osculating plane

$$n \cdot \left(x-0, y-1, z-\frac{\pi}{2}\right) = 0$$

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{2}\right) = 0 \Longrightarrow \boxed{x+z-\frac{\pi}{2} = 0}$$